大阪大学大学院基礎工学研究科

乱流力学特論

非線形力学領域熱工学グループ

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講義内容（1）

0．導入：乱流に対する本講義の視点
　乱流の統計的性質及び構造の動力学
1．非圧縮性流体の運動
　1.1 変形速度と渦度
　1.2 渦度方程式
　1.3 非粘性保存量
2．渦層及び渦管の動力学
　2.1 伸長拡散渦層，渦管
　2.2 渦線の巻き込み

講義内容（2）

3．一様等方乱流及び壁近傍乱流の普遍統計法則
　3.1 Kolmogorov の4/5則
　3.2 普遍相似則
　　Kolmogorov の相似則，－5/3乗則
　　Prandtl の壁法則，対数則
4．剪断乱流における秩序構造の動力学
　4.1 縦渦の動力学
　K., Kida, Tanaka, Yanase *JFM* 353 (1997)
　K. *Phys. Fluids* 17 (2005)

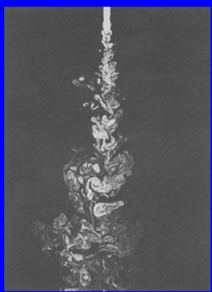
講義内容（3）

4．剪断乱流における秩序構造の動力学
　4.2 ストリークの不安定（縦渦の生成）
　K., Jimenez, Uhlmann, Pinelli *JFM* 483 (2003)
5．乱流遷移及び乱流への力学系的アプローチ
　5.1 不安定周期軌道による壁近傍乱流の記述
　K., Kida *JFM* 449 (2001)
　K., Kida, Nagata, *IUTAM Symp. One Hundred Years of Boundary Layer Research* (2006)

講義内容（4）

5．乱流遷移及び乱流への力学系的アプローチ
　5.2 不安定周期軌道による壁近傍乱流の制御
　Jimenez, K., Simens, Nagata, Shiba, *Phys. Fluids* 17 (2005)
　K., *Phys. Fluids* 17 (2005)
　5.3 不安定周期軌道による等方乱流の記述
　van Veen, Kida, K., *FDR* 38 (2006)

0．導入
乱流の統計的性質及び構造の動力学

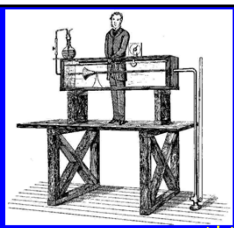


Dimotakis *et al.* (1981)

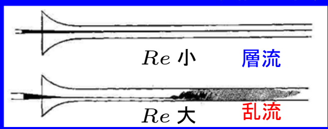
Higuchi (1993)

乱流の発生

Osborne Reynolds
(1842 – 1912)



$Re = \dfrac{Ud}{\nu}$　U:流速　d:管径　ν:動粘性係数



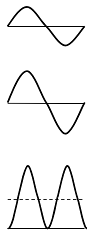
Reynolds (1883)

乱流現象
非線形性, エネルギー散逸, 巨大自由度

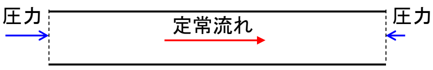
$$y = \sin x$$

$$ay + by = (a+b)\sin x$$

$$ay \times by = \frac{ab}{2} - \frac{ab}{2}\cos 2x$$

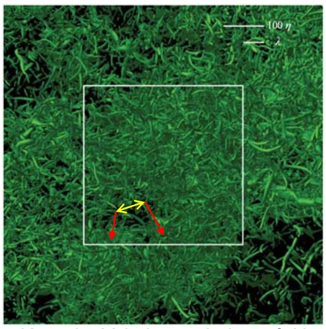


乱流現象
非線形性, **エネルギー散逸**, 巨大自由度



力学的エネルギー非保存
圧力による仕事率＝エネルギー散逸率
粘性（摩擦）による発熱
運動エネルギー⇒内部エネルギー

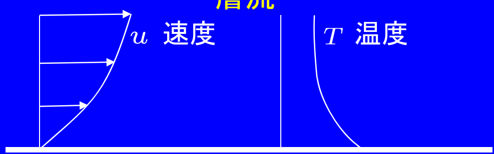
乱流現象
非線形性, エネルギー散逸, **巨大自由度**



Ishihara, Kaneda, Yokokawa, Itakura & Uno (2007)

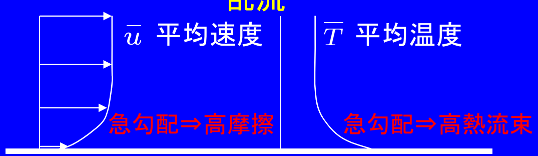
運動量・熱輸送

層流



滑りなし，加熱壁面

乱流



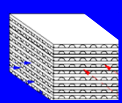
滑りなし，加熱壁面

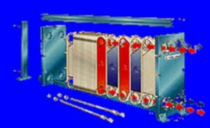
工学的応用

流動抵抗低減

航空機やパイプライン流体輸送
などにおける摩擦抵抗低減

伝熱促進





コンパクト，プレート形熱交換器
などの伝熱性能向上

乱流現象

非線形性，エネルギー散逸，巨大自由度

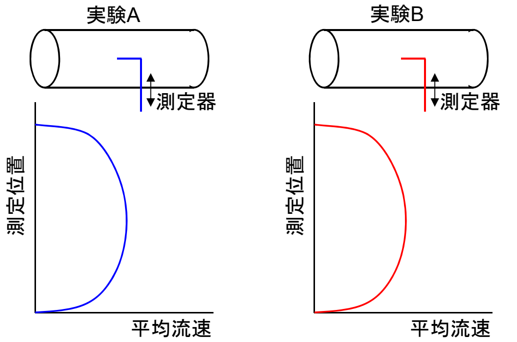
乱流構造の動力学

非再現性

乱流統計量

未完結性

乱流統計量
平均流速（統計量）の再現性



完結問題

$$\frac{du}{dt} = -u^2, \quad u(t=0) = u_0$$

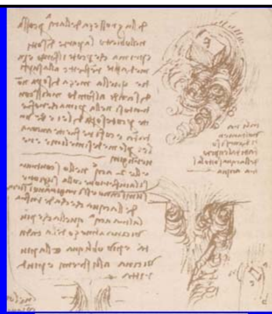
$$u = \frac{u_0}{1 + u_0 t}$$

$$\frac{d\overline{u}}{dt} = -\overline{u^2}, \quad \frac{d\overline{u^2}}{dt} = -2\overline{u^3}, \quad \cdots$$

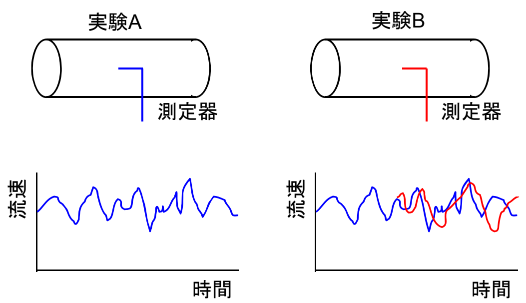
Davidson (2004)

秩序構造

Leonardo da Vinci
(1452 – 1519)



瞬時流速の非再現性



カオス（初期値敏感性）

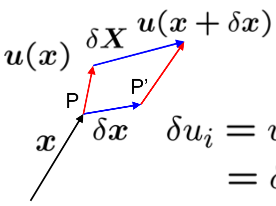
- 乱流の統計的性質
  再現性あり
  支配方程式は閉じない
- 乱流（秩序構造，渦構造）の動力学
  再現性なし
  支配方程式は単純

参考書

- 「流体力学(前編)」
  今井　功，裳華房
- 「乱流力学」
  木田重雄，柳瀬眞一郎，朝倉書店
- 「乱流理論の基礎」
  後藤俊幸，朝倉書店
- Turbulence
  P. A. Davidson, Oxford University Press

1．非圧縮性流体の運動

1.1 変形速度と渦度



流体の相対運動
2点P，P'の速度差

$$\delta u_i \quad (i=1,2,3)$$

$$\delta u_i = u_i(\boldsymbol{x}+\delta\boldsymbol{x}) - u_i(\boldsymbol{x})$$
$$= \delta X_i - \delta x_i$$
$$= \frac{\partial u_i}{\partial x_j}\delta x_j + O(\delta x^2)$$

$$\delta x = |\delta \boldsymbol{x}|$$

$\delta x \to 0$

$$\delta u_i = \frac{\partial u_i}{\partial x_j}\delta x_j$$

速度勾配テンソル(velocity-gradient tensor)

$$D = \{d_{i,j}\} = \left\{\frac{\partial u_i}{\partial x_j}\right\}$$

テンソル T の固有多項式(eigen-polynomial)

$$F(\lambda) = -\det(T - \lambda I)$$
$$= \lambda^3 - \mathrm{tr}(T)\lambda^2 + Q(T)\lambda - \det(T)$$

テンソル第2不変量

$$Q(T) = \tfrac{1}{2}\left\{[\mathrm{tr}(T)]^2 - \mathrm{tr}(T^2)\right\}$$

非圧縮流体

$$\mathrm{tr}(D) = d_{i,i} = \frac{\partial u_i}{\partial x_i} = \nabla\cdot\boldsymbol{u} = 0$$
$$Q(D) = \tfrac{1}{2}[(d_{i,i})^2 - d_{i,j}d_{j,i}]$$
$$= -\frac{1}{2}\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

$Q(D)$ は乱流中の管状渦構造
の可視化に用いられる

速度勾配テンソルの分解

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{E=\{e_{i,j}\}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\Omega=\{\omega_{i,j}\}}$$

変形速度テンソル
rate-of-strain tensor

$$e_{i,j} = e_{j,i}$$

渦度テンソル
vorticity tensor

$$\omega_{i,j} = -\omega_{j,i}$$

変形速度テンソルで表される相対速度場

$$\delta\boldsymbol{u} = \tfrac{1}{2}E\delta\boldsymbol{x}$$

座標系の回転，反転　直交行列 A $(A^{-1} = A^t)$

$$\delta\boldsymbol{u} = A\delta\boldsymbol{u}' \qquad \delta\boldsymbol{x} = A\delta\boldsymbol{x}'$$

$$A\delta\boldsymbol{u}' = \tfrac{1}{2}EA\delta\boldsymbol{x}' \Rightarrow \delta\boldsymbol{u}' = \tfrac{1}{2}A^tEA\delta\boldsymbol{x}'$$

$$\tfrac{1}{2}A^tEA = \begin{pmatrix} \frac{\partial u'_1}{\partial x'_1} & & 0 \\ & \frac{\partial u'_2}{\partial x'_2} & \\ 0 & & \frac{\partial u'_3}{\partial x'_3} \end{pmatrix}$$

非圧縮流体

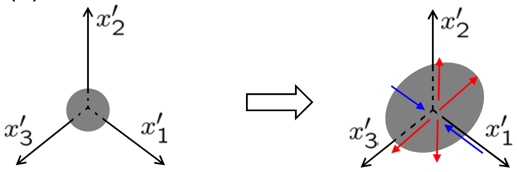
$$\mathrm{tr}(E) = 2\mathrm{tr}(D) = 2\frac{\partial u_i}{\partial x_i} = 2\frac{\partial u'_i}{\partial x'_i} = 0$$

$$\frac{\partial u'_1}{\partial x'_1} \le \frac{\partial u'_2}{\partial x'_2} \le \frac{\partial u'_3}{\partial x'_3}$$

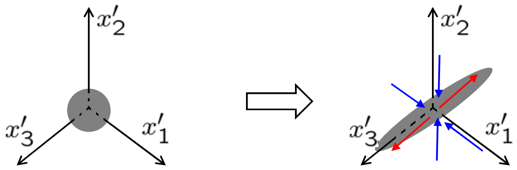
$$(1)\ \frac{\partial u'_1}{\partial x'_1} \le 0 \le \frac{\partial u'_2}{\partial x'_2} \le \frac{\partial u'_3}{\partial x'_3}$$

$$(2)\ \frac{\partial u'_1}{\partial x'_1} \le \frac{\partial u'_2}{\partial x'_2} \le 0 \le \frac{\partial u'_3}{\partial x'_3}$$

(1) の場合



(2) の場合



渦度テンソルで表される相対速度場

$$\delta\boldsymbol{u} = \tfrac{1}{2}\Omega\delta\boldsymbol{x}$$

$$\omega_{i,j} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = (\delta_{i,l}\delta_{j,m} - \delta_{i,m}\delta_{j,l})\frac{\partial u_l}{\partial x_m}$$

$$= \varepsilon_{j,i,k}\varepsilon_{k,m,l}\frac{\partial u_l}{\partial x_m} = \varepsilon_{j,i,k}\omega_k$$

渦度（vorticity）

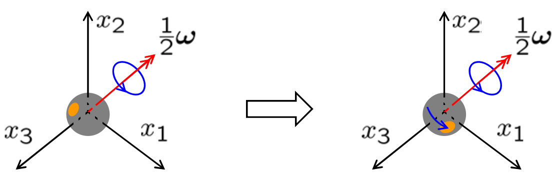
$$\omega_k = \varepsilon_{k,m,l}\frac{\partial u_l}{\partial x_m}$$

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$$

$$\delta u_i = \tfrac{1}{2}\omega_{i,j}\delta x_j$$

$$= \tfrac{1}{2}\epsilon_{j,i,k}\omega_k\delta x_j = \tfrac{1}{2}\epsilon_{i,k,j}\omega_k\delta x_j$$

$$\delta\boldsymbol{u} = \tfrac{1}{2}\Omega\delta\boldsymbol{x} = \tfrac{1}{2}\boldsymbol{\omega} \times \delta\boldsymbol{x}$$



局所的相対運動

$$\delta\boldsymbol{u} = D\delta\boldsymbol{x} = \tfrac{1}{2}E\delta\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{\omega} \times \delta\boldsymbol{x}$$

直交する3方向への一様な伸縮　　一様な回転

$$Q(D) = -\tfrac{1}{2}d_{i,j}d_{j,i}$$

$$= -\tfrac{1}{8}(e_{i,j} + \omega_{i,j})(e_{i,j} - \omega_{i,j})$$

$$= \tfrac{1}{8}(\|\Omega\|^2 - \|E\|^2)$$

$Q(D) > 0$ 回転　　$Q(D) < 0$ 伸縮

1.2 渦度方程式

連続の式（質量保存則）

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{u} = 0$$

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla$$

初期時刻に密度一定であれば，それ以後も密度一定に保たれる

以下では，密度一定の流れを考える

非圧縮Newton流体の応力テンソル

$$P = \{-p\delta_{i,j} + \mu e_{i,j}\}$$

Navier-Stokes方程式（運動方程式）

$$\rho\frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial}{\partial x_j}[-p\delta_{i,j} + \mu e_{i,j}]$$

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\nu = \frac{\mu}{\rho}$$

Navier-Stokes方程式の不変性

1. 時間並進不変性
2. 空間並進不変性
3. 空間回転，反転不変性
4. Galilei変換不変性
5. スケール変換不変性

$$\boldsymbol{x}' = \lambda \boldsymbol{x},\ \ \boldsymbol{u}' = \lambda^{\alpha/3}\boldsymbol{u},$$
$$t' = \lambda^{1-\alpha/3}t,\ \ \rho' = \rho,$$
$$p' = \lambda^{2\alpha/3}p,\ \ \nu' = \lambda^{1+\alpha/3}\nu$$

Navier-Stokes $\alpha = -3$
Euler α 任意 Kolmogorov $\alpha = 1$

Navier-Stokes方程式の発散

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_j \partial x_j} = -\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} - \left(\frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j} - \nu\frac{\partial^2}{\partial x_j \partial x_j}\right)\frac{\partial u_i}{\partial x_i}$$

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_j \partial x_j} = -\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} = 2Q(D)$$

$$\nabla^2 p = 2\rho Q(D)$$

圧力の非局所性

$$p(\boldsymbol{x}) = -\frac{\rho}{2\pi}\iiint_V \frac{Q(D')}{|\boldsymbol{x} - \boldsymbol{x}'|}\mathrm{d}V' + \phi(\boldsymbol{x})$$

$$\nabla^2 \phi = 0$$

Navier-Stokes方程式の回転

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}|\boldsymbol{u}|^2\right) + \boldsymbol{u}\times\boldsymbol{\omega} + \nu\nabla^2\boldsymbol{u}$$

$$\Longrightarrow\ \frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}) + \nu\nabla^2\boldsymbol{\omega}$$

$$\frac{\mathrm{D}\omega_i}{\mathrm{D}t} = \underline{\omega_j\frac{\partial u_i}{\partial x_j}} + \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

渦度伸長項（vorticity-stretching term）

$$\omega_j\frac{\partial u_i}{\partial x_j} = \omega_j\left(\tfrac{1}{2}e_{i,j} + \tfrac{1}{2}\omega_{i,j}\right)$$
$$= \tfrac{1}{2}e_{i,j}\omega_j + \tfrac{1}{2}\varepsilon_{i,k,j}\omega_k\omega_j$$

$$D\boldsymbol{\omega} = \tfrac{1}{2}E\boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\omega}\times\boldsymbol{\omega}$$
$$= \tfrac{1}{2}E\boldsymbol{\omega}$$

渦度方程式（vorticity equation）

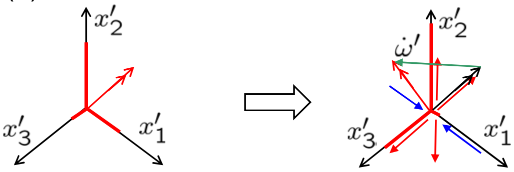
$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \tfrac{1}{2}E\boldsymbol{\omega} + \nu\nabla^2\boldsymbol{\omega}$$

$$\frac{\mathrm{D}\omega_i}{\mathrm{D}t} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\omega_j + \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

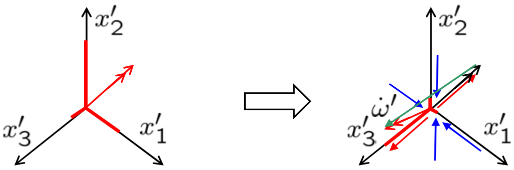
$$\dot{\boldsymbol{\omega}} = \tfrac{1}{2}E\boldsymbol{\omega}$$

$$\dot{\boldsymbol{\omega}}' = \tfrac{1}{2}A^t E A\boldsymbol{\omega}'$$

(1) の場合



(2) の場合



実質線素の方程式

$$\frac{\mathrm{D}\delta\boldsymbol{\ell}}{\mathrm{D}t} = D\delta\boldsymbol{\ell}$$

$$\frac{\mathrm{D}\delta\boldsymbol{\ell}}{\mathrm{D}t} = \tfrac{1}{2}E\delta\boldsymbol{\ell} + \tfrac{1}{2}\boldsymbol{\omega}\times\delta\boldsymbol{\ell}$$

Helmholtzの渦定理

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \tfrac{1}{2}E\boldsymbol{\omega}$$

$$\delta\boldsymbol{\ell} \parallel \boldsymbol{\omega} \to \boldsymbol{\omega} = a\delta\boldsymbol{\ell}$$

$$\frac{\mathrm{D}a}{\mathrm{D}t}\delta\boldsymbol{\ell} + a\frac{\mathrm{D}\delta\boldsymbol{\ell}}{\mathrm{D}t} = \tfrac{1}{2}aE\delta\boldsymbol{\ell} \to \frac{\mathrm{D}a}{\mathrm{D}t} = 0$$

変形速度テンソルの時間発展

$$\frac{\mathrm{D}E}{\mathrm{D}t} = -\tfrac{1}{2}(E^2 + \Omega^2) - \frac{2}{\rho}\Pi + \nu\nabla^2 E$$

圧力Hessian

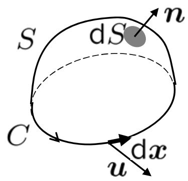
$$\Pi = \left\{\frac{\partial^2 p}{\partial x_i \partial x_j}\right\}$$

渦度テンソルの時間発展

$$\frac{\mathrm{D}\Omega}{\mathrm{D}t} = -\tfrac{1}{2}(E\Omega + \Omega E) + \nu\nabla^2\Omega$$

1.3 非粘性保存量

局所保存量：循環（circulation）の保存



$$\Gamma(C) = \oint_C \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x} = \iint_S \boldsymbol{\omega}\cdot\boldsymbol{n}\mathrm{d}S$$

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \oint_C \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\cdot\mathrm{d}\boldsymbol{x} + \boldsymbol{u}\cdot\frac{\mathrm{D}\mathrm{d}\boldsymbol{x}}{\mathrm{D}t}\right)$$

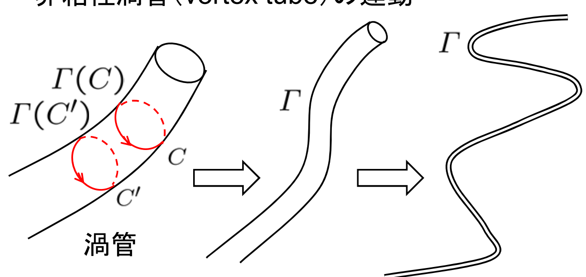
$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\boldsymbol{u} \qquad \frac{\mathrm{D}\mathrm{d}\boldsymbol{x}}{\mathrm{D}t} = (\mathrm{d}\boldsymbol{x}\cdot\nabla)\boldsymbol{u}$$

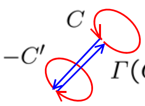
$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \nu\oint_C (\nabla^2\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{x} \Rightarrow$$

Kelvinの循環定理

$\nu = 0 \to \Gamma =$ 一定

非粘性渦管（vortex tube）の運動





$\Gamma(C - C') = 0$

$\Gamma(C) + \Gamma(-C') = 0$

$\Gamma(C) - \Gamma(C') = 0$

$\Gamma = \omega\delta S$

$\delta S \to 0,\ \omega \to \infty$

Biot-Savart則

$$\boldsymbol{u} = \nabla\times\boldsymbol{A} + \nabla\phi \qquad \nabla\cdot\boldsymbol{A} = 0, \quad \nabla^2\phi = 0$$

$$\nabla^2\boldsymbol{A} = -\nabla\times\boldsymbol{u} = -\boldsymbol{\omega}$$

$$\boldsymbol{A} = \frac{1}{4\pi}\iiint_V \frac{\boldsymbol{\omega}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}V'$$

$$\nabla\times\boldsymbol{A} = \frac{1}{4\pi}\iiint_V \frac{\boldsymbol{\omega}(\boldsymbol{x}')\times(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\mathrm{d}V'$$

$$\approx \frac{\Gamma}{4\pi}\int_\ell \frac{\mathrm{d}\boldsymbol{\ell}'\times(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}$$

大域保存量
運動エネルギーの保存

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}|\boldsymbol{u}|^2\right) = -\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}|\boldsymbol{u}|^2 u_j + \frac{p}{\rho}u_j\right) + \nu u_i \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\begin{aligned} u_i\frac{\partial^2 u_i}{\partial x_j \partial x_j} &= \frac{\partial}{\partial x_j}(u_i d_{i,j}) - \|D\|^2 \\ &= \frac{\partial}{\partial x_j}(u_i e_{i,j}) - \tfrac{1}{2}\|E\|^2 \\ &= \frac{\partial}{\partial x_j}(u_i \omega_{i,j}) - |\boldsymbol{\omega}|^2 \\ &= \nabla\cdot(\boldsymbol{u}\times\boldsymbol{\omega}) - |\boldsymbol{\omega}|^2 \end{aligned}$$

空間平均 $\overline{(\,)}^V = \dfrac{1}{V}\displaystyle\iiint_V (\,)\mathrm{d}V$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\overline{|\boldsymbol{u}|^2}^V\right) = \begin{cases} -\nu\overline{\|D\|^2}^V \\ -\nu\tfrac{1}{2}\overline{\|E\|^2}^V \\ -\nu\overline{|\boldsymbol{\omega}|^2}^V \end{cases} = -2\nu Q$$

エンストロフィー
(enstrophy) $Q = \tfrac{1}{2}\overline{|\boldsymbol{\omega}|^2}^V$

$\nu = 0$ $\quad \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\overline{|\boldsymbol{u}|^2}^V\right) = 0$ エネルギーカスケード

大域保存量
エンストロフィーの保存

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}|\omega|^2\right) = -\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}|\omega|^2 u_j\right) + \frac{1}{2}\omega_i e_{i,j}\omega_j + \nu\omega_i\frac{\partial^2\omega_i}{\partial x_j\partial x_j}$$

$$\omega_i\frac{\partial^2\omega_i}{\partial x_j\partial x_j} = \nabla\cdot[\boldsymbol{\omega}\times(\nabla\times\boldsymbol{\omega})] - |\nabla\times\boldsymbol{\omega}|^2$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \tfrac{1}{2}\overline{\boldsymbol{\omega}^t E\boldsymbol{\omega}}^V$$
⇦ 渦度伸長による
エンストロフィー変化

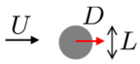
$$+\frac{\nu}{V}\iint_{\partial V}\frac{\partial}{\partial n}\left(\tfrac{1}{2}|\boldsymbol{\omega}|^2\right)\mathrm{d}S - \nu\overline{|\nabla\times\boldsymbol{\omega}|^2}^V$$

$\nu = 0$ $\quad \dfrac{\mathrm{d}Q}{\mathrm{d}t} = \tfrac{1}{2}\overline{\boldsymbol{\omega}^t E\boldsymbol{\omega}}^V$ エンストロフィー非保存

2次元流 $\quad \dfrac{\mathrm{d}Q}{\mathrm{d}t} = 0$ エンストロフィーカスケード

非粘性エネルギー散逸

$\nu \to 0 \quad Q \to \infty \quad \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\overline{|\boldsymbol{u}|^2}^V\right) = -2\nu Q < 0$

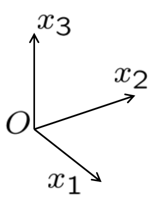


$$C_D = \frac{D}{\tfrac{1}{2}\rho U^2 L^2}$$

$$\dot{W} = DU = \tfrac{1}{2}\rho U^3 L^2 C_D \qquad \epsilon \sim \frac{\dot{W}}{\rho L^3} \sim \frac{U^3}{L}C_D$$

2. 渦層及び渦管の動力学

2.1 伸長拡散渦層，渦管
拡散渦層



2次元流

$$\boldsymbol{u} = u_1(x_1,x_2,t)\boldsymbol{e}_1 + u_2(x_1,x_2,t)\boldsymbol{e}_2$$

$$\boldsymbol{\omega} = \omega_3(x_1,x_2,t)\boldsymbol{e}_3 = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\boldsymbol{e}_3$$

1方向流 $\quad u_2 \equiv 0$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = \frac{\partial u_1}{\partial x_1} = 0$$

$$u_1 = u_1(x_2,t)$$

$$\omega_3 = -\frac{\partial u_1}{\partial x_2} = \omega_3(x_2,t)$$

$$\frac{\partial\omega_3}{\partial t} = \nu\frac{\partial^2\omega_3}{\partial x_2^2}$$

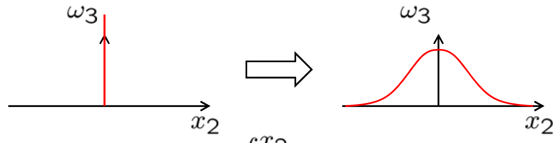
相似解　$\omega_3(t=0)=2U_0\delta(x_2)$

$$\omega_3(x_2,t)\propto\frac{U_0}{\sqrt{\nu t}}f\Bigg(\underbrace{\frac{x_2}{\sqrt{\nu t}}}_{=\xi\ \text{相似変数}}\Bigg)$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}+\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\tfrac{1}{2}\xi f\right)=0$$

対称解　$f(\xi)\propto \mathrm{e}^{-\xi^2/4}$

$$\Rightarrow\quad \omega_3=\frac{1}{\sqrt{\pi}}\frac{U_0}{\sqrt{\nu t}}\exp\left(-\frac{x_2^2}{4\nu t}\right)$$



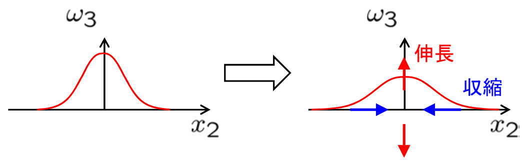
速度場　$u_1=-\int_0^{x_2}\omega_3(x_2',t)\mathrm{d}x_2'$

$$=-\frac{2U_0}{\sqrt{\pi}}\mathrm{erf}\left(\frac{x_2}{2\sqrt{\nu t}}\right)$$

$$\mathrm{erf}(x)=\int_0^x \mathrm{e}^{-s^2}\mathrm{d}s\quad \mathrm{erf}(\pm\infty)=\pm\frac{\sqrt{\pi}}{2}$$

循環密度 $\int_{-\infty}^{+\infty}\omega_3\mathrm{d}x_2=-[u_1]_{-\infty}^{+\infty}=2U_0$

伸長拡散渦層



2次元非圧縮渦なしひずみ流

$$\boldsymbol{u}=u_1(x_2,t)\boldsymbol{e}_1-\gamma x_2\boldsymbol{e}_2+\gamma x_3\boldsymbol{e}_3$$

$$\boldsymbol{\omega}=\omega_3(x_2,t)\boldsymbol{e}_3\quad=-\frac{\partial u_1}{\partial x_2}\boldsymbol{e}_3$$

定常解(Burgers渦層)　$\omega_3(x_2)$

$$\frac{\partial\omega_3}{\partial t}-\gamma x_2\frac{\partial\omega_3}{\partial x_2}=\gamma\omega_3+\nu\frac{\partial^2\omega_3}{\partial x_2^2}$$

$$\nu\frac{\mathrm{d}^2\omega_3}{\mathrm{d}x_2^2}+\gamma\frac{\mathrm{d}}{\mathrm{d}x_2}(x_2\omega_3)=0$$

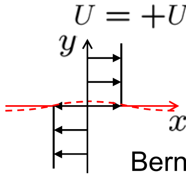
対称解

$$\omega_3=\sqrt{\frac{2}{\pi}}\frac{U_0}{\sqrt{\nu/\gamma}}\exp\left(-\frac{x_2^2}{2\nu/\gamma}\right)$$

$$u_1=-\frac{2U_0}{\sqrt{\pi}}\mathrm{erf}\left(\frac{x_2}{\sqrt{2\nu/\gamma}}\right)$$

渦層の不安定(Kelvin-Helmholtz不安定)

$U=+U_0$　　渦層(厚さ無限小)



$U=-U_0$

$$\boldsymbol{u}=U\boldsymbol{e}_x+\nabla\phi$$

$$\nabla^2\phi=0$$

Bernoulli方程式　$\rho=1$

$$\frac{\partial\phi}{\partial t}+\tfrac{1}{2}|\boldsymbol{u}|^2+(P+p)=f(t)$$

線形化方程式(無限遠で撹乱は減衰)

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\phi+p=0$$

渦層(実質面)　$S(x,y)+s(x,y,t)=0$

$$\frac{\mathrm{D}}{\mathrm{D}t}(S+s)=0$$

線形化方程式　$S(x,y)\equiv y=0$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)s+\frac{\partial\phi}{\partial y}\frac{\partial S}{\partial y}=0$$

ノーマル・モード　$f=\phi,\ p,\ s$

$$f(x,y,t)=\mathrm{Re}\left[\tilde{f}(y)\,\mathrm{e}^{\mathrm{i}\alpha(x-ct)}\right]$$

撹乱方程式

$$\frac{\mathrm{d}^2\tilde{\phi}}{\mathrm{d}y^2} - \alpha^2\tilde{\phi} = 0$$

$$\mathrm{i}\alpha(U - c)\tilde{\phi} + \tilde{p} = 0$$

$$\mathrm{i}\alpha(U - c)\tilde{s} + \frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}y} = 0 \quad (y = 0)$$

$$\tilde{\phi}(y) = \begin{cases} \tilde{\phi}^+(y) & (y > 0) \\ \tilde{\phi}^-(y) & (y < 0) \end{cases}$$

接合条件 $(y = 0)$

$$(U_0 - c)\tilde{\phi}^+ = (-U_0 - c)\tilde{\phi}^-$$

$$(-U_0 - c)\frac{\mathrm{d}\tilde{\phi}^+}{\mathrm{d}y} = (U_0 - c)\frac{\mathrm{d}\tilde{\phi}^-}{\mathrm{d}y}$$

固有解

$$\tilde{\phi}^+ = A\mathrm{e}^{-\alpha y} \qquad \tilde{\phi}^- = B\mathrm{e}^{+\alpha y}$$

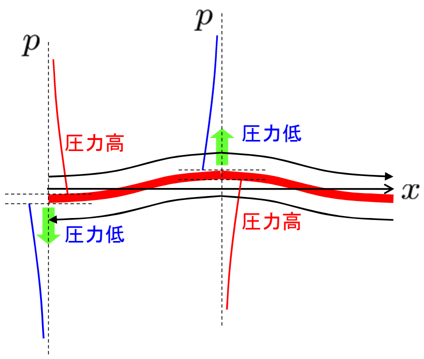
$$(U_0 - c)A + (U_0 + c)B = 0$$

$$(U_0 + c)A - (U_0 - c)B = 0$$

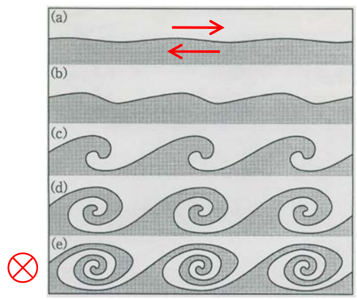
$$c/U_0 = \pm\mathrm{i} \Longrightarrow \alpha\mathrm{Im}(c) = \pm\alpha U_0$$

$$B/A = \pm\mathrm{i}$$

不安定化のメカニズム

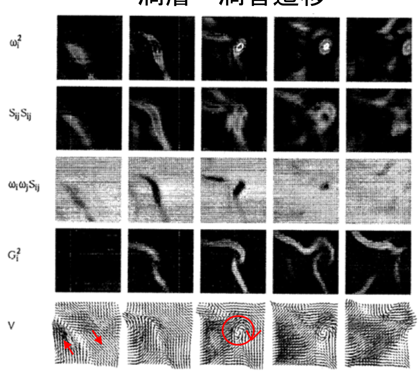


渦層の巻き上がり



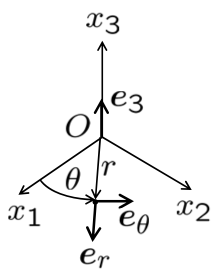
Krasny (1986)

渦層ー渦管遷移



Ruetsch & Maxey (1992)

拡散渦管



2次元流

$$\boldsymbol{u} = u_r(r, \theta, t)\boldsymbol{e}_r + u_\theta(r, \theta, t)\boldsymbol{e}_\theta$$

$$\boldsymbol{\omega} = \omega_3(x_1, x_2, t)\boldsymbol{e}_3$$

$$\frac{\partial \boldsymbol{e}_r}{\partial r} = 0 \qquad \frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta$$

$$\frac{\partial \boldsymbol{e}_\theta}{\partial r} = 0 \qquad \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r$$

円形流 $u_r \equiv 0$

$$\nabla_\perp \cdot \boldsymbol{u} = \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \cdot (u_r \boldsymbol{e}_r + u_\theta \boldsymbol{e}_\theta)$$
$$= \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0$$
$$\frac{\partial u_\theta}{\partial \theta} = 0$$
$$u_\theta = u_\theta(r, t)$$

$$\boldsymbol{\omega} = \nabla_\perp \times \boldsymbol{u}$$
$$= \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \times (u_\theta \boldsymbol{e}_\theta)$$
$$= \frac{\partial u_\theta}{\partial r}\boldsymbol{e}_r \times \boldsymbol{e}_\theta - \frac{u_\theta}{r}\boldsymbol{e}_\theta \times \boldsymbol{e}_r$$
$$= \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\boldsymbol{e}_3$$
$$\omega_3 = \omega_3(r, t) = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)$$

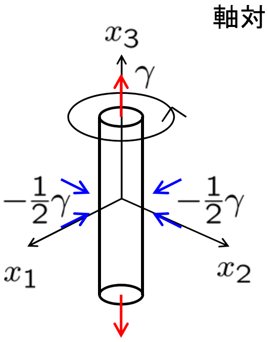
$$\boldsymbol{u} \cdot \nabla_\perp = (u_r \boldsymbol{e}_r + u_\theta \boldsymbol{e}_\theta) \cdot \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right)$$
$$= u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}$$
$$\nabla_\perp^2 = \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \cdot \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right)$$
$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

2次元流
$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla_\perp)\boldsymbol{\omega} = \nu \nabla_\perp^2 \boldsymbol{\omega}$$
$$\frac{\partial \omega_3}{\partial t} + \left(u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\right)\omega_3$$
$$= \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\omega_3$$
円形流
$$\frac{\partial \omega_3}{\partial t} = \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\omega_3$$

相似解 $\omega_3(t=0) = \varGamma \delta(r)/(\pi r)$
$$\omega_3(r, t) \propto \frac{\varGamma}{\nu t} f\left(\underbrace{\frac{r}{\sqrt{\nu t}}}_{=\,\xi}\right)$$
$= \xi$ 相似変数
$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi \frac{\mathrm{d}f}{\mathrm{d}\xi}\right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\tfrac{1}{2}\xi^2 f\right) = 0$$
正則解（Lamb-Oseen渦）$f(\xi) \propto \mathrm{e}^{-\xi^2/4}$
$$\Longrightarrow \omega_3 = \frac{\varGamma}{4\pi\nu t}\exp\left(-\frac{r^2}{4\nu t}\right)$$

速度場
$$u_\theta = \frac{1}{r}\int_0^r r' \omega_3(r', t)\mathrm{d}r'$$
$$= \frac{\varGamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right]$$
循環（半径 r の円周に沿う）
$$\varGamma(r; t) = \int_0^{2\pi} u_\theta r \mathrm{d}\theta = \varGamma\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right]$$
$$\varGamma(r = \infty; t) = \varGamma$$

伸長拡散渦管



軸対称非圧縮渦なしひずみ流

$$\boldsymbol{u} = u_\theta(r,t)\boldsymbol{e}_\theta - \frac{\gamma}{2} r \boldsymbol{e}_r + \gamma x_3 \boldsymbol{e}_3$$

$$\boldsymbol{\omega} = \omega_3(r,t)\boldsymbol{e}_3 = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\boldsymbol{e}_3$$

定常解（Burgers渦管） $\omega_3(r)$

$$\frac{\partial \omega_3}{\partial t} - \frac{\gamma}{2} r \frac{\partial \omega_3}{\partial r} = \gamma \omega_3 + \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) \omega_3$$

$$\nu \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}\omega_3}{\mathrm{d}r} \right) + \frac{\gamma}{2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \omega_3 \right) = 0$$

正則解

$$\omega_3 = \frac{\Gamma}{4\pi\nu/\gamma} \exp\left(-\frac{r^2}{4\nu/\gamma} \right)$$

$$u_\theta = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu/\gamma} \right) \right]$$

非定常解 $\omega_3(r,t)$

$$\frac{\partial \omega_3}{\partial t} - \frac{\gamma}{2} r \frac{\partial \omega_3}{\partial r} = \gamma \omega_3 + \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) \omega_3$$

Lundgren (1982) の変換
収縮，伸長効果を非定常項に繰り込む

伸長パラメター $S(t)$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \gamma S \qquad S(t=0) = 1$$

$$S = \exp\left(\int_0^t \gamma(t')\mathrm{d}t' \right) = \mathrm{e}^{\gamma t}$$

$$R = [S(t)]^{1/2} r = \mathrm{e}^{\gamma t/2} r$$

$$T = T_0 + \int_0^t S(t')\mathrm{d}t' = T_0 + (\mathrm{e}^{\gamma t} - 1)/\gamma$$

$$\Omega(R,T) = \omega_3(r,t)/S(t) = \omega_3 \mathrm{e}^{-\gamma t}$$

$$\frac{\partial}{\partial t} = \frac{\partial T}{\partial t}\frac{\partial}{\partial T} + \frac{\partial R}{\partial t}\frac{\partial}{\partial R} = S\frac{\partial}{\partial T} + \frac{\gamma}{2} R \frac{\partial}{\partial R}$$

$$\frac{\partial}{\partial r} = \frac{\partial T}{\partial r}\frac{\partial}{\partial T} + \frac{\partial R}{\partial r}\frac{\partial}{\partial R} = S^{1/2}\frac{\partial}{\partial R}$$

$$\frac{\partial^2}{\partial r^2} = S\frac{\partial^2}{\partial R^2}$$

渦度方程式

$$\frac{\partial}{\partial t}(S\Omega) - \frac{\gamma}{2} r \frac{\partial}{\partial r}(S\Omega) = \gamma S\Omega + \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) S\Omega$$

$$\gamma S\Omega + S\left(S\frac{\partial}{\partial T} + \frac{\gamma}{2} R \frac{\partial}{\partial R} \right)\Omega - \frac{\gamma}{2} S R \frac{\partial \Omega}{\partial R} = \gamma S\Omega + \nu S^2 \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} \right) \Omega$$

$$\frac{\partial \Omega}{\partial T} = \nu \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} \right) \Omega$$

Lamb-Oseen渦

$$\Omega = \frac{\Gamma}{4\pi\nu T} \exp\left(-\frac{R^2}{4\nu T} \right)$$

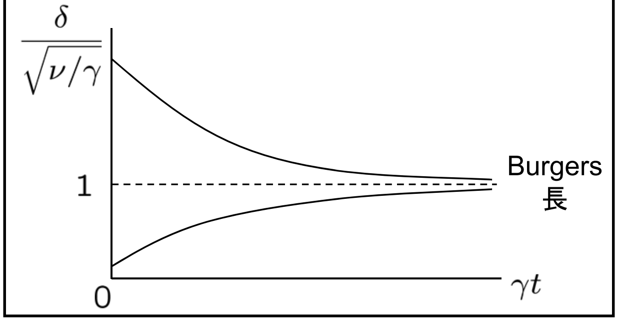
伸長拡散渦

$$\omega_3 = S\Omega = \frac{\Gamma}{4\pi\delta^2} \exp\left(-\frac{r^2}{4\delta^2} \right)$$

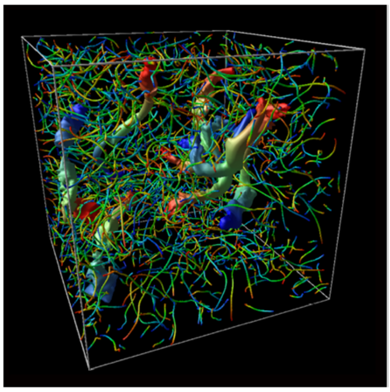
$$u_\theta = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\delta^2} \right) \right]$$

伸長拡散渦の太さ

$$\delta(t) = \sqrt{\nu/\gamma + (\nu T_0 - \nu/\gamma)\mathrm{e}^{-\gamma t}}$$



等方乱流の基本秩序構造



渦管

直径

$10(\nu^3/\bar{\epsilon})^{1/4}$

速度

$3(\nu\bar{\epsilon})^{1/4}$

エネルギー散逸率

$\bar{\epsilon} = \nu\overline{\|E\|^2}/2$

Kida & Miura (2000)

乱流の小スケールにおける渦管

$$\text{Burgers 長} = \sqrt{\nu/\gamma} \sim (\nu^3/\bar{\epsilon})^{1/4}$$

$$\text{ひずみ時間スケール} \quad \frac{1}{\gamma} \sim \sqrt{\nu/\bar{\epsilon}}$$

$$\text{Burgers 速度} = \frac{\Gamma}{\sqrt{\nu/\gamma}} \sim (\nu\bar{\epsilon})^{1/4}$$

$$\text{渦Reynolds数} \quad \frac{\Gamma}{\nu} \sim 1$$

$$\text{旋回時間スケール} \quad \frac{\nu}{\Gamma}\frac{1}{\gamma} \sim \sqrt{\nu/\bar{\epsilon}}$$

壁近傍乱流の基本秩序構造

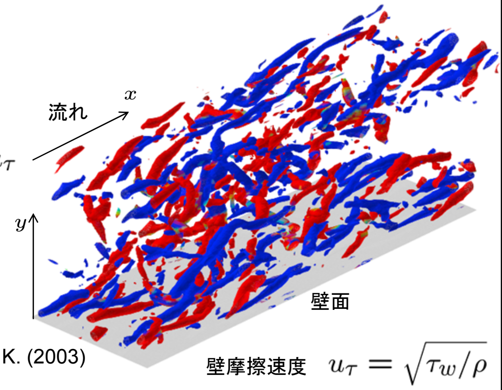
渦管

直径

$30\nu/u_\tau$

速度

u_τ



K. (2003)

壁摩擦速度 $u_\tau = \sqrt{\tau_w/\rho}$

壁近傍乱流における縦渦

$$\text{Burgers 長} = \sqrt{\nu/\gamma} \sim \nu/u_\tau$$

$$\text{ひずみ時間スケール} \quad \frac{1}{\gamma} \sim \nu/u_\tau^2$$

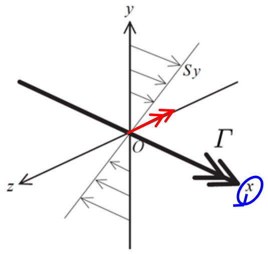
$$\text{Burgers 速度} = \frac{\Gamma}{\sqrt{\nu/\gamma}} \sim u_\tau$$

$$\text{渦Reynolds数} \quad \frac{\Gamma}{\nu} \sim 1$$

$$\text{旋回時間スケール} \quad \frac{\nu}{\Gamma}\frac{1}{\gamma} \sim \nu/u_\tau^2$$

2.2 渦線の巻き込み：流れ方向渦の動力学

広義2次元流



渦度

$$\boldsymbol{\omega} = \omega_x(y,z,t)\boldsymbol{e}_x + \omega_y(y,z,t)\boldsymbol{e}_y + \omega_z(y,z,t)\boldsymbol{e}_z$$

速度

$$\boldsymbol{u} = u(y,z,t)\boldsymbol{e}_x + v(y,z,t)\boldsymbol{e}_y + w(y,z,t)\boldsymbol{e}_z$$

流れ関数

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\Rightarrow \quad v=\frac{\partial \psi}{\partial z} \qquad w=-\frac{\partial \psi}{\partial y}$$

$$\omega_x=-\nabla_\perp^2\psi=-\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi$$

$$\omega_y=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=\frac{\partial u}{\partial z}$$

$$\omega_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=-\frac{\partial u}{\partial y}$$

渦度

$$\boldsymbol{\omega}(y,z,t)=-\nabla_\perp^2\psi\,\boldsymbol{e}_x+\frac{\partial u}{\partial z}\boldsymbol{e}_y-\frac{\partial u}{\partial y}\boldsymbol{e}_z$$

速度

$$\boldsymbol{u}(y,z,t)=u\,\boldsymbol{e}_x+\frac{\partial \psi}{\partial z}\boldsymbol{e}_y-\frac{\partial \psi}{\partial y}\boldsymbol{e}_z$$

渦線の(y,z)面への射影

$$\left(\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)(\omega_y,\omega_z)^t=\left(\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)\left(\frac{\partial u}{\partial z},-\frac{\partial u}{\partial y}\right)^t=0$$

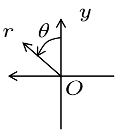
$$\nabla_\perp u\perp(\omega_y,\omega_z) \quad \Rightarrow \quad u=\text{const.}$$

渦線の射影

渦度方程式

$$\frac{\partial \omega_x}{\partial t}+v\frac{\partial \omega_x}{\partial y}+w\frac{\partial \omega_x}{\partial z}=\nu\nabla_\perp^2\omega_x$$

初期条件：$\omega_x(t=0)=\dfrac{\Gamma\delta(r)}{\pi r}$



拡散渦管　$(\xi=\frac{1}{2}r(\nu t)^{-1/2})$

$$\omega_x(r,t)=\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)=\frac{\Gamma}{4\pi\nu t}\mathrm{e}^{-\xi^2}$$

$$u_\theta(r,t)=\frac{\Gamma}{2\pi r}\left(1-\mathrm{e}^{-\xi^2}\right)$$

Navier-Stokes方程式

$$\frac{\partial u}{\partial t}+\frac{u_\theta}{r}\frac{\partial u}{\partial \theta}=\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)u$$

初期条件：$u(t=0)=Sy=Sr\cos\theta$

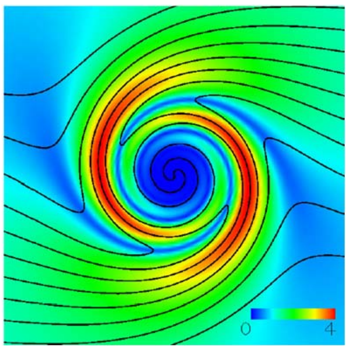
相似解　$u(r,\theta,t)=Sr\,\mathrm{Re}[f(\xi)\,\mathrm{e}^{-\mathrm{i}\theta}]$

$$f''+\left(2\xi+\frac{3}{\xi}\right)f'+\mathrm{i}\frac{\Gamma}{2\pi\nu}\frac{1-\mathrm{e}^{-\xi^2}}{\xi^2}f=0$$

境界条件：$f(\xi=\infty)=1$

Pearson-Abernathy (1984); Moore (1985)
らせん渦層

y



z

渦軸垂直渦度

$$\frac{\sqrt{\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2}}{S}$$

$$\frac{\Gamma}{2\pi\nu}=100$$

$40\sqrt{\nu t}$

3. 一様等方乱流及び壁近傍乱流の普遍統計法則

3.1 Kolmogorov の4/5則

集団平均　$\overline{x}=\displaystyle\int_{-\infty}^{\infty}xf(x)\mathrm{d}x$

$$\overline{x}=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}x_i$$

時間平均　$\overline{x}^T=\displaystyle\lim_{T\to\infty}\frac{1}{T}\int_0^T x\mathrm{d}t$

空間平均　$\overline{x}^V=\displaystyle\lim_{V\to\infty}\frac{1}{V}\int_V x\mathrm{d}V$

エルゴード性（ergodicity）

非圧縮 Navier–Stokes 方程式 ($\rho = 1$)

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

Reynolds 分解

$$u_i = \overline{u}_i + u_i', \quad p = \overline{p} + p'$$

平均方程式

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\nu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) \underbrace{- \overline{u_i' u_j'}}_{\text{Reynolds応力}}\right]$$

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

乱れ方程式

$$\frac{\partial u_i'}{\partial t} + (\overline{u}_j + u_j')\frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\overline{u_i' u_j'} = -\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_j \partial x_j}$$

$$\frac{\partial u_i'}{\partial x_i} = 0$$

乱れエネルギー方程式

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}|\boldsymbol{u}'|^2\right) + \frac{\partial}{\partial x_j}\left[(\overline{u}_j + u_j')\tfrac{1}{2}|\boldsymbol{u}'|^2\right]$$
$$+ u_i' u_j' \frac{\partial \overline{u}_i}{\partial x_j} - u_i' \frac{\partial}{\partial x_j}\overline{u_i' u_j'} = -\frac{\partial}{\partial x_i}(p' u_i') + \nu u_i' \frac{\partial^2 u_i'}{\partial x_j \partial x_j}$$

平均乱れエネルギー方程式

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\overline{|\boldsymbol{u}'|^2}\right) = -\frac{\partial}{\partial x_j}\left[\overline{(\overline{u}_j + u_j')\tfrac{1}{2}|\boldsymbol{u}'|^2} - \overline{(-p'\delta_{i,j} + \nu e_{i,j}')u_i'}\right]$$
$$-\frac{\nu}{2}\overline{e_{i,j}' e_{i,j}'} - \overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j}$$

$$e_{i,j}' = \frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}$$

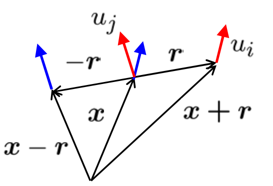
エネルギー散逸

$$\boxed{\text{流れ}} \xrightarrow{\frac{\nu}{2} e_{i,j} e_{i,j} = \nu e_{i,j} \frac{\partial u_i}{\partial x_j}} \boxed{\text{分子運動}}$$

乱れエネルギー生成

$$\boxed{\text{平均流}} \xleftrightarrow{-\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j}} \boxed{\text{乱流運動}}$$

一様乱流の2次速度相関テンソル



$$\begin{aligned} R_{i,j}(\boldsymbol{r}) &= \overline{u_i(\boldsymbol{x}+\boldsymbol{r}) u_j(\boldsymbol{x})} \\ &= \overline{u_i(\boldsymbol{x}) u_j(\boldsymbol{x}-\boldsymbol{r})} \\ &= \overline{u_i(\boldsymbol{x}) u_j(\boldsymbol{x}')} \\ &= R_{i,j}(\boldsymbol{x}, \boldsymbol{x}') \end{aligned}$$

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{r}$$

$$\overline{u_\alpha(\boldsymbol{x}+\boldsymbol{r}) u_\beta(\boldsymbol{x})} = R_{i,j}\alpha_i \beta_j$$

$$\frac{\partial}{\partial x_k} R_{i,j}(\boldsymbol{x}, \boldsymbol{x}') = \frac{\partial}{\partial r_k} R_{i,j}(\boldsymbol{r}) = -\frac{\partial}{\partial x_k'} R_{i,j}(\boldsymbol{x}, \boldsymbol{x}')$$

非圧縮条件

$$\begin{aligned} \frac{\partial}{\partial r_i} R_{i,j}(\boldsymbol{r}) &= \frac{\partial}{\partial x_i} R_{i,j}(\boldsymbol{x}, \boldsymbol{x}') \\ &= \overline{\frac{\partial u_i}{\partial x_i}(\boldsymbol{x}) u_j(\boldsymbol{x}')} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial r_j} R_{i,j}(\boldsymbol{r}) &= -\frac{\partial}{\partial x_j'} R_{i,j}(\boldsymbol{x}, \boldsymbol{x}') \\ &= -\overline{u_i(\boldsymbol{x}) \frac{\partial u_j}{\partial x_j}(\boldsymbol{x}')} \\ &= 0 \end{aligned}$$

一様等方乱流の2次速度相関テンソル

$$R_{i,j}(\boldsymbol{r}) = F(r)\frac{r_i r_j}{r^2} + G(r)\delta_{i,j}$$

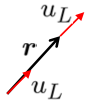
$$R_{i,j}(\boldsymbol{r}) = R_{i,j}(-\boldsymbol{r}) = R_{j,i}(\boldsymbol{r})$$

RMS(root-mean-square)速度

$$U^2 = \tfrac{1}{3}R_{i,i}(0) = \tfrac{1}{3}\overline{|\boldsymbol{u}|^2}$$

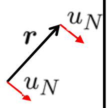
2次縦速度相関関数

$$f(r) = \overline{u_L(\boldsymbol{x}+\boldsymbol{r})u_L(\boldsymbol{x})}/U^2$$



2次横速度相関関数

$$g(r) = \overline{u_N(\boldsymbol{x}+\boldsymbol{r})u_N(\boldsymbol{x})}/U^2$$



縦速度相関

$$U^2 f(r) = R_{i,j}(\boldsymbol{r})\frac{r_i r_j}{r^2} = F(r) + G(r)$$

横速度相関

$$\boldsymbol{u}_N = \boldsymbol{u} - \frac{\boldsymbol{u}\cdot\boldsymbol{r}}{r}\frac{\boldsymbol{r}}{r}$$

$$u_{Nk} = \left(\delta_{k,i} - \frac{r_k r_i}{r^2}\right)u_i$$

$$U^2 g(r) = \tfrac{1}{2}\left(\delta_{k,i} - \frac{r_k r_i}{r^2}\right)\left(\delta_{k,j} - \frac{r_k r_j}{r^2}\right)\overline{u_i(\boldsymbol{x}+\boldsymbol{r})u_j(\boldsymbol{x})}$$

$$= \tfrac{1}{2}\left(\delta_{i,j} - \frac{r_i r_j}{r^2}\right)R_{i,j}(\boldsymbol{r}) = G(r)$$

$$F(r) = U^2[f(r) - g(r)] \qquad G(r) = U^2 g(r)$$

非圧縮条件

$$\frac{\partial}{\partial r_i}R_{i,j}(\boldsymbol{r}) = \frac{\partial}{\partial x_i}R_{i,j}(\boldsymbol{x},\boldsymbol{x}')$$

$$= U^2\left[\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{2}{r}(f-g)\right]\frac{r_j}{r}$$

$$= 0$$

$$\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{2}{r}(f-g) = 0$$

$$g = f + \frac{r}{2}\frac{\mathrm{d}f}{\mathrm{d}r}$$

一様性

$$R_{1,1}(r_1\boldsymbol{e}_1) = \overline{u_1(\boldsymbol{x}+r_1\boldsymbol{e}_1)u_1(\boldsymbol{x})}$$

$$\frac{\partial}{\partial x_1}R_{1,1}(\boldsymbol{x},\boldsymbol{x}') = -\frac{\partial}{\partial x_1'}R_{1,1}(\boldsymbol{x},\boldsymbol{x}')$$

$$\frac{\partial}{\partial x_1}R_{1,1}(\boldsymbol{x},\boldsymbol{x}) = -\frac{\partial}{\partial x_1}R_{1,1}(\boldsymbol{x},\boldsymbol{x}) = 0$$

Taylor長

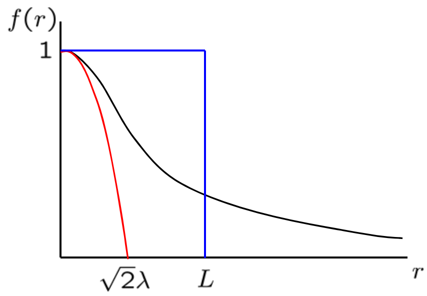
$$\lambda = \left(-\left.\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}\right|_{r=0}\right)^{-1/2}$$

$$f(r) = 1 - \tfrac{1}{2}\left(\frac{r}{\lambda}\right)^2 + \cdots$$

$$g(r) = 1 - \left(\frac{r}{\lambda}\right)^2 + \cdots$$

積分長

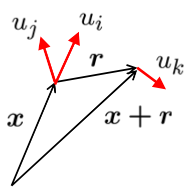
$$L = \int_0^\infty f(r)\mathrm{d}r \qquad \int_0^\infty g(r)\mathrm{d}r = \tfrac{1}{2}L$$



一様等方乱流の3次速度相関テンソル

$$R_{i,j,k}(\boldsymbol{r}) = \overline{u_i(\boldsymbol{x})u_j(\boldsymbol{x})u_k(\boldsymbol{x}+\boldsymbol{r})}$$

$$R_{i,j,k}(\boldsymbol{r}) = R_{j,i,k}(\boldsymbol{r})$$



$$\frac{\partial}{\partial r_k}R_{i,j,k}(\boldsymbol{r})$$

$$= \overline{u_i(\boldsymbol{x})u_j(\boldsymbol{x})\frac{\partial u_k}{\partial x_k}(\boldsymbol{x}+\boldsymbol{r})} = 0$$

$$R_{i,j,k}(\boldsymbol{r}) = A(r)\frac{r_i r_j r_k}{r^3} + B(r)\delta_{i,j}\frac{r_k}{r} + C(r)\left(\delta_{i,k}\frac{r_j}{r} + \delta_{j,k}\frac{r_i}{r}\right)$$

$$R_{i,j,k}(\boldsymbol{r}) = -R_{i,j,k}(-\boldsymbol{r}) \qquad R_{i,j,k}(0) = 0$$

非圧縮条件

$$\frac{\partial}{\partial r_k}R_{i,j,k}(\boldsymbol{r}) = \left[\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{2}{r}\right)A + 2\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{1}{r}\right)C\right]\frac{r_i r_j}{r^2} + \left[\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{2}{r}\right)B + \frac{2}{r}C\right]\delta_{i,j} = 0$$

$$C = -B - \frac{r}{2}\frac{\mathrm{d}B}{\mathrm{d}r}$$

$$A = -B + r\frac{\mathrm{d}B}{\mathrm{d}r}$$

3次縦速度相関関数

$$h(r) = \overline{u_L^2(\boldsymbol{x})u_L(\boldsymbol{x}+\boldsymbol{r})}/U^3$$

縦速度相関

$$U^3 h(r) = R_{i,j,k}\frac{r_i r_j r_k}{r^3} = -2B(r)$$

等方性

$$R_{1,1,1}(r_1\boldsymbol{e}_1) = -R_{1,1,1}(-r_1\boldsymbol{e}_1)$$

$$\frac{\partial^2}{\partial r_1^2}R_{1,1,1}(r_1\boldsymbol{e}_1) = -\frac{\partial^2}{\partial r_1^2}R_{1,1,1}(-r_1\boldsymbol{e}_1)$$

$$\frac{\partial^2}{\partial r_1^2}R_{1,1,1}(0) = -\frac{\partial^2}{\partial r_1^2}R_{1,1,1}(0) = 0$$

一様性 $\frac{\partial}{\partial r_1}R_{1,1,1}(0) = \overline{u_1^2\frac{\partial u_1}{\partial x_1}} = \frac{1}{3}\frac{\partial}{\partial x_1}\overline{u_1^3} = 0$

$$\begin{aligned}\frac{\partial^3}{\partial r_1^3}R_{1,1,1}(0) &= \overline{u_1^2\frac{\partial^3 u_1}{\partial x_1^3}}\\ &= \frac{\partial}{\partial x_1}\overline{u_1^2\frac{\partial^2 u_1}{\partial x_1^2}} - \overline{\frac{\partial u_1^2}{\partial x_1}\frac{\partial^2 u_1}{\partial x_1^2}}\\ &= -\overline{2u_1\frac{\partial u_1}{\partial x_1}\frac{\partial^2 u_1}{\partial x_1^2}} \qquad = -\overline{u_1\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}\right)^2}\\ &= -\frac{\partial}{\partial x_1}\overline{u_1\left(\frac{\partial u_1}{\partial x_1}\right)^2} + \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3}\\ &= \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3}\end{aligned}$$

縦速度相関

$$U^3 h(r) = \frac{1}{6}\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3}r^3 + \cdots$$

2次速度構造テンソル

$$\begin{aligned}B_{i,j}(\boldsymbol{r}) &= \overline{\delta u_i(\boldsymbol{r})\delta u_j(\boldsymbol{r})}\\ &= 2[R_{i,j}(0) - R_{i,j}(\boldsymbol{r})]\\ &= S_2(r)\frac{r_i r_j}{r^2} + U_2(r)\left(\delta_{i,j} - \frac{r_i r_j}{r^2}\right)\end{aligned}$$

$$\delta u_i(\boldsymbol{r}) = u_i(\boldsymbol{x}+\boldsymbol{r}) - u_i(\boldsymbol{x})$$

2次縦速度構造関数

$$\begin{aligned}S_2(r) &= \overline{[u_L(\boldsymbol{x}+\boldsymbol{r}) - u_L(\boldsymbol{x})]^2}\\ &= 2U^2[1 - f(r)]\end{aligned}$$

2次横速度構造関数

$$\begin{aligned}U_2(r) &= \overline{[u_N(\boldsymbol{x}+\boldsymbol{r}) - u_N(\boldsymbol{x})]^2}\\ &= 2U^2\left[1 - f(r) - \frac{r}{2}\frac{\mathrm{d}f}{\mathrm{d}r}\right]\\ &= S_2(r) + \frac{r}{2}\frac{\mathrm{d}S_2}{\mathrm{d}r}\end{aligned}$$

3次速度構造テンソル

$$\begin{aligned}B_{i,j,k}(\boldsymbol{r}) &= \overline{\delta u_i(\boldsymbol{r})\delta u_j(\boldsymbol{r})\delta u_k(\boldsymbol{r})}\\ &= 2[R_{i,j,k}(\boldsymbol{r}) + R_{k,i,j}(\boldsymbol{r}) + R_{j,k,i}(\boldsymbol{r})]\\ &= U^3\Big[3\left(h - r\frac{\mathrm{d}h}{\mathrm{d}r}\right)\frac{r_i r_j r_k}{r^3}\\ &\quad + \left(h + r\frac{\mathrm{d}h}{\mathrm{d}r}\right)\left(\delta_{i,j}\frac{r_k}{r} + \delta_{j,k}\frac{r_i}{r} + \delta_{k,i}\frac{r_j}{r}\right)\Big]\end{aligned}$$

3次縦速度構造関数

$$\begin{aligned}S_3(r) &= \overline{[u_L(\boldsymbol{x}+\boldsymbol{r}) - u_L(\boldsymbol{x})]^3}\\ &= B_{i,j,k}(\boldsymbol{r})\frac{r_i r_j r_k}{r^3}\\ &= 6U^3 h(r)\end{aligned}$$

速度相関方程式

$$u_i = u_i(\boldsymbol{x}, t) \qquad u_i' = u_i(\boldsymbol{x}', t)$$
$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{r}$$
$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k} u_i u_k = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2}{\partial x_k \partial x_k} u_i + f_i$$
$$\frac{\partial u_j'}{\partial t} + \frac{\partial}{\partial x_k'} u_j' u_k' = -\frac{\partial p'}{\partial x_j'} + \nu \frac{\partial^2}{\partial x_k' \partial x_k'} u_j' + f_j'$$
$$\frac{\partial}{\partial t} \overline{u_i u_j'} = -\frac{\partial}{\partial r_k} \left(\overline{u_i u_k u_j'} - \overline{u_i u_k' u_j'} \right)$$
$$-\frac{\partial}{\partial r_i} \overline{p u_j'} + \frac{\partial}{\partial r_j} \overline{p' u_i} + 2\nu \frac{\partial^2}{\partial r_k \partial r_k} \overline{u_i u_j'} + \overline{f_i u_j'} + \overline{f_j' u_i}$$

$$\frac{\partial}{\partial t} R_{i,j}(\boldsymbol{r}, t) = \frac{\partial}{\partial r_k} \left[R_{i,k,j}(\boldsymbol{r}, t) + R_{j,k,i}(\boldsymbol{r}, t) \right]$$
$$-\frac{\partial}{\partial r_i} \overline{p u_j'} + \frac{\partial}{\partial r_j} \overline{p' u_i} + 2\nu \frac{\partial^2}{\partial r_k \partial r_k} R_{i,j}(\boldsymbol{r}, t) + F_{i,j}$$

圧力速度相関

$$\overline{p u_j'} = A(r, t) \frac{r_j}{r}$$
$$\frac{\partial}{\partial r_j} \overline{p u_j'} = \frac{\partial}{\partial r_j} \left(A \frac{r_j}{r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A) = 0$$
$$A(r, t) = \frac{C}{r^2}$$
$$\overline{p u_j'} = 0 \quad (C = 0)$$

$$\left(\frac{\partial}{\partial t} - 2\nu \frac{\partial^2}{\partial r_k \partial r_k} \right) R_{i,j}(\boldsymbol{r}, t)$$
$$= \frac{\partial}{\partial r_k} \left[R_{i,k,j}(\boldsymbol{r}, t) + R_{j,k,i}(\boldsymbol{r}, t) \right] + F_{i,j}$$

3次速度相関項

$$\frac{\partial}{\partial r_k} \left[R_{i,k,j}(\boldsymbol{r}, t) + R_{j,k,i}(\boldsymbol{r}, t) \right] \Big|_{r=0}$$
$$= \overline{u_i u_k \frac{\partial u_j}{\partial x_k}} + \overline{u_j u_k \frac{\partial u_i}{\partial x_k}}$$
$$= \frac{\partial}{\partial x_k} \overline{u_i u_j u_k} = 0$$

等方表現 $i = j$

$$\left(3 + r \frac{\partial}{\partial r} \right) \left[\frac{\partial}{\partial t} U^2 f - 2\nu U^2 \left(\frac{\partial}{\partial r} + \frac{4}{r} \right) \frac{\partial f}{\partial r} \right.$$
$$\left. - U^3 \left(\frac{\partial}{\partial r} + \frac{4}{r} \right) h - F(r, t) \right] = 0$$
$$F(r, t) = \frac{1}{r^3} \int_0^r r'^2 F_{i,i}(r', t) \mathrm{d}r'$$

Karman—Howarth方程式

$$\left[\frac{\partial}{\partial t} - 2\nu \left(\frac{\partial}{\partial r} + \frac{4}{r} \right) \frac{\partial}{\partial r} \right] U^2 f = \left(\frac{\partial}{\partial r} + \frac{4}{r} \right) U^3 h + F$$

エネルギー散逸率

減衰乱流　$F(r, t) = 0$

$$\frac{\partial}{\partial t} U^2 = 10 \nu U^2 \frac{\partial^2 f}{\partial r^2} \Big|_{r=0}$$
$$\bar{\epsilon} = -\frac{\mathrm{d}}{\mathrm{d}t} \left(\tfrac{3}{2} U^2 \right) = 15 \nu \left(\frac{U}{\lambda} \right)^2$$

定常乱流　$\partial / \partial t = 0$

$$F(r = 0) = -10 \nu U^2 \frac{\partial^2 f}{\partial r^2} \Big|_{r=0}$$
$$\bar{\epsilon} = \tfrac{3}{2} F(r = 0) = 15 \nu \left(\frac{U}{\lambda} \right)^2$$

積分長

$$\frac{U^3}{L} \sim \nu \left(\frac{U}{\lambda} \right)^2$$
$$\frac{L}{\lambda} \sim \frac{U \lambda}{\nu} = R_\lambda$$
$$Re = \frac{UL}{\nu} \sim \frac{U\lambda}{\nu} \frac{L}{\lambda} = R_\lambda^2$$

Kolmogorov長

$$\eta = (\nu^3 / \bar{\epsilon})^{1/4}$$
$$\frac{\eta}{\lambda} \sim R_\lambda^{-1/2}$$
$$\frac{\eta}{L} \sim R_\lambda^{-3/2} \sim Re^{-3/4}$$

2，3次縦速度相関

$$U^2 f(r,t) = U^2 - \tfrac{1}{2}S_2(r,t)$$
$$U^3 h(r,t) = \tfrac{1}{6}S_3(r,t)$$

Karman—Howarth方程式

$$\frac{\mathrm{d}U^2}{\mathrm{d}t} - \frac{1}{2}\frac{\partial S_2}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\left(\tfrac{1}{6}S_3 - \nu\frac{\partial S_2}{\partial r}\right)\right] + F$$

小スケール　$r \ll L$

$$\tfrac{1}{6}S_3 - \nu\frac{\partial S_2}{\partial r} = -\tfrac{2}{15}\bar{\epsilon}r$$

慣性領域（inertial range）$\eta \ll r \ll L$

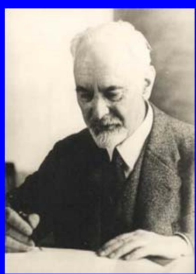
$$\tfrac{1}{6}S_3 = -\tfrac{2}{15}\bar{\epsilon}r$$

Kolmogorovの4/5則

$$S_3 = \overline{[u_L(\boldsymbol{x}+\boldsymbol{r},t) - u_L(\boldsymbol{x},t)]^3} = -\tfrac{4}{5}\bar{\epsilon}r$$
$$U^3 h = \overline{u_L^2(\boldsymbol{x},t)u_L(\boldsymbol{x}+\boldsymbol{r},t)} = -\tfrac{2}{15}\bar{\epsilon}r$$

3.2 普遍相似則

Andrei N. Kolmogorov
(1903 – 1987)
Kolmogorov の相似則
(1941)



Ludwig Prandtl
(1875 – 1953)
Prandtl の壁法則
(1932)

Fourier展開 L^3 周期箱

$$\boldsymbol{u}(\boldsymbol{x},t) = \left(\frac{2\pi}{L}\right)^3 \sum_{\boldsymbol{k}} \tilde{\boldsymbol{u}}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$$
$$L \to \infty \qquad \sum_{\boldsymbol{k}}\left(\frac{2\pi}{L}\right)^3 \to \int \mathrm{d}\boldsymbol{k}$$

波数ベクトル

$$\boldsymbol{k} = \frac{2\pi}{L}\boldsymbol{m} \qquad \boldsymbol{m} = (m_1, m_2, m_3)^t \qquad m_i : \text{整数}$$

Fourier係数

$$\tilde{\boldsymbol{u}}(\boldsymbol{k},t) = \left(\frac{1}{2\pi}\right)^3 \int \boldsymbol{u}(\boldsymbol{x},t)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{x}$$
$$\tilde{\boldsymbol{u}}(-\boldsymbol{k},t) = \tilde{\boldsymbol{u}}^*(\boldsymbol{k},t)$$

エネルギースペクトルテンソル

$$\Phi_{i,j}(\boldsymbol{k},t) = \left(\frac{1}{2\pi}\right)^3 \int R_{i,j}(\boldsymbol{r},t)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$$
$$R_{i,j}(\boldsymbol{r},t) = \int \Phi_{i,j}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{k}$$

エネルギースペクトル関数

$$\tfrac{1}{2}\overline{|\boldsymbol{u}|^2} = \tfrac{3}{2}U^2 = \tfrac{1}{2}R_{i,i}(0,t) = \tfrac{1}{2}\int \Phi_{i,i}(\boldsymbol{k},t)\mathrm{d}\boldsymbol{k}$$
$$= \int_0^\infty 2\pi k^2 \Phi_{i,i}(k,t)\mathrm{d}k \qquad k = |\boldsymbol{k}|$$
$$= \int_0^\infty E(k,t)\mathrm{d}k$$

渦度相関テンソル

$$R^{\omega}_{i,j}(\boldsymbol{r}) = \overline{\omega_i(\boldsymbol{x}+\boldsymbol{r})\omega_j(\boldsymbol{x})} = \overline{\omega_i(\boldsymbol{x})\omega_j(\boldsymbol{x}')}$$
$$R^{\omega}_{i,i}(\boldsymbol{r}) = \varepsilon_{i,p,q}\varepsilon_{i,l,m}\overline{\frac{\partial u_q}{\partial x_p}\frac{\partial u'_m}{\partial x'_l}}$$
$$= \overline{\frac{\partial u_m}{\partial x_l}\frac{\partial u'_m}{\partial x'_l}} - \overline{\frac{\partial u_l}{\partial x_m}\frac{\partial u'_m}{\partial x'_l}}$$
$$= \frac{\partial^2}{\partial x_l \partial x'_l}R_{m,m} - \frac{\partial^2}{\partial x_m \partial x'_l}R_{l,m}$$
$$= -\frac{\partial^2}{\partial r_l^2}R_{m,m} + \frac{\partial^2}{\partial r_m \partial r_l}R_{l,m}$$
$$= -\frac{\partial^2}{\partial r_l^2}R_{m,m}$$

エンストロフィースペクトルテンソル

$$\Phi^{\omega}_{i,j}(\boldsymbol{k},t)=\left(\frac{1}{2\pi}\right)^{3}\int R^{\omega}_{i,j}(\boldsymbol{r},t)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$$

$$R^{\omega}_{i,j}(\boldsymbol{r},t)=\int \Phi^{\omega}_{i,j}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{k}$$

エンストロフィースペクトル関数

$$R^{\omega}_{i,i}(\boldsymbol{r},t)=-\frac{\partial^2}{\partial r_l^2}\int \Phi_{i,i}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{k}$$

$$=\int k^2\Phi_{i,i}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{k}$$

$$\frac{1}{2}\overline{|\boldsymbol{\omega}|^2}=\frac{1}{2}R^{\omega}_{i,i}(0,t)=\frac{1}{2}\int k^2\Phi_{i,i}(\boldsymbol{k},t)\mathrm{d}\boldsymbol{k}$$

$$=\int_0^\infty 2\pi k^4\Phi_{i,i}(k,t)\mathrm{d}k=\int_0^\infty k^2E(k,t)\mathrm{d}k$$

エネルギー散逸

$$2\nu\frac{\partial^2}{\partial r_k^2}R_{i,i}=-2\nu R^{\omega}_{i,i}$$

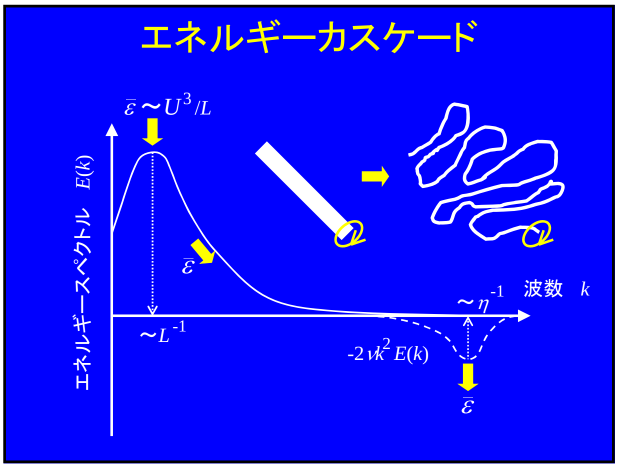
$$2\nu\left.\frac{\partial^2}{\partial r_k^2}R_{i,i}\right|_{r=0}=-2\nu R^{\omega}_{i,i}(0,t)=-2\nu\overline{|\boldsymbol{\omega}|^2}$$

$$3\frac{\mathrm{d}U^2}{\mathrm{d}t}+2\nu\overline{|\boldsymbol{\omega}|^2}=F_{i,i}(r=0)$$

減衰乱流 $\bar{\epsilon}=-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2}U^2\right)=\nu\overline{|\boldsymbol{\omega}|^2}$

定常乱流 $\bar{\epsilon}=\frac{1}{2}F_{i,i}(r=0)=\nu\overline{|\boldsymbol{\omega}|^2}$

エネルギー散逸スペクトル $2\nu k^2E(k,t)$



Kolmogorov の相似則

エネルギー散逸

$$\bar{\epsilon}=\frac{\nu}{2}\overline{e_{i,j}e_{i,j}}\qquad e_{i,j}=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}$$

Kolmogorov スケール

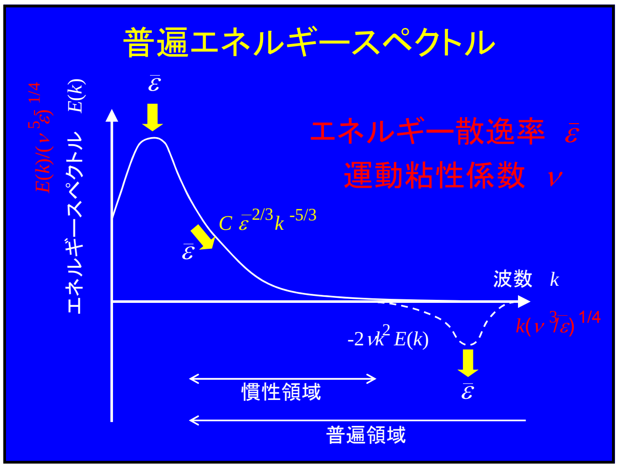
$$\eta=(\nu^3/\bar{\epsilon})^{1/4},\ t_K=(\nu/\bar{\epsilon})^{1/2},\ u_K=(\nu\bar{\epsilon})^{1/4}$$

エネルギースペク トル関数（第 1 仮説）

$$E(k)=(\nu^5\bar{\epsilon})^{1/4}f[k(\nu^3/\bar{\epsilon})^{1/4}]$$

$-5/3$ 乗法則（第 2 仮説）

$$E(k)=C_K\bar{\epsilon}^{2/3}k^{-5/3}\qquad C_K=1.4-1.8$$



Kolmogorov カスケード

慣性領域 $\bar{\epsilon},\ r$

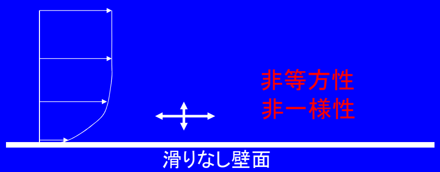
$$\delta u(r)\sim(\bar{\epsilon}r)^{1/3}$$

速度 $\dfrac{\delta u(r)}{\delta u(r')}\sim\left(\dfrac{r}{r'}\right)^{1/3}$ スケール変換不変 $\alpha=1$

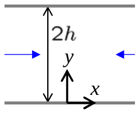
渦度 $\dfrac{\omega(r)}{\omega(r')}\sim\left(\dfrac{r}{r'}\right)^{-2/3}$

間欠性 (intermittency) $\overline{[\delta u(r)]^p}/U^p\sim(r/L)^{p/3}$ からのずれ

壁近傍乱流



- 境界層流
- 二次元チャネル流 (Couette, Poiseuille)
- 円管流 (Hagen-Poiseuille)



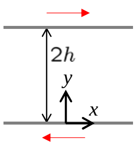
平面 Poiseuille 流

$$\overline{p} + \overline{v'^2} = \overline{p}|_w$$

$$\nu \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - \overline{u'v'} = \nu \left.\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right|_w - \frac{\partial \overline{p}}{\partial x} y$$

平面 Couette 流 $(\partial\overline{p}/\partial x = 0)$

$$\nu \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - \overline{u'v'} = \nu \left.\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right|_w$$



平面 Poiseuille 流

$$\overline{p} + \overline{v'^2} = \overline{p}|_w$$

$$\nu \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - \overline{u'v'} = \nu \left.\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right|_w - \frac{\partial \overline{p}}{\partial x} y$$

平面 Couette 流 $(\partial\overline{p}/\partial x = 0)$

$$\nu \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - \overline{u'v'} = \nu \left.\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right|_w$$

壁面摩擦速度

$$u_\tau = \sqrt{\nu \left|\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right|_w}$$

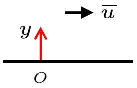
Prandtl の壁法則（壁近傍領域 $y/h \ll 1$）

$$\nu \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - \overline{u'v'} = u_\tau^2$$

$$\overline{u} = u_\tau f\left(\frac{u_\tau y}{\nu}\right)$$

$$\frac{\overline{u}}{u_\tau} = \frac{u_\tau y}{\nu} \quad (u_\tau y/\nu \sim 1)$$

平均速度分布



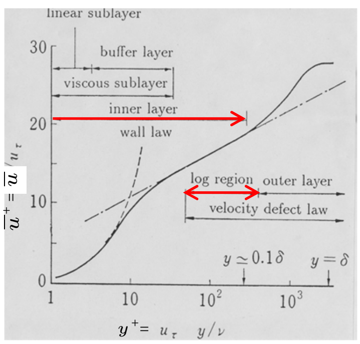
運動粘性係数

ν

壁摩擦速度

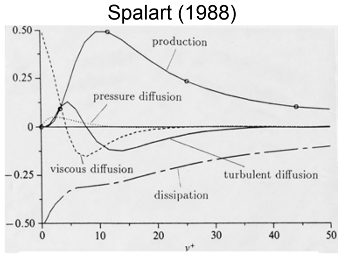
$$u_\tau = \sqrt{\nu \left|\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right|_w}$$

$$\nu/u_\tau \sim (\nu^3/\overline{\epsilon})^{1/4}$$



乱流エネルギー収支

Spalart (1988)



$$-\frac{\mathrm{d}\phi_2}{\mathrm{d}y} + \nu \frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\overline{v_i'v_i'}/2\right) - \overline{u'v'}\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - \epsilon = 0$$

× 2

$$\phi_2 = \overline{v'(p' + v_i'v_i'/2)}, \quad \epsilon = \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

対数速度分布（Townsend 1976）

乱流エネルギー生成

$$u_\tau^2 \frac{\mathrm{d}\overline{u}}{\mathrm{d}y}$$

エネルギー散逸 ～ エネルギー注入

$$\epsilon \sim \frac{u_\tau^3}{\kappa y}$$

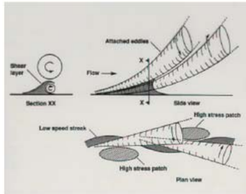
エネルギー生成 ～ エネルギー散逸

$$\frac{\mathrm{d}\overline{u}^+}{\mathrm{d}y^+} \sim \frac{1}{\kappa y^+} \to \overline{u}^+ \sim \frac{1}{\kappa}\ln y^+ + C$$

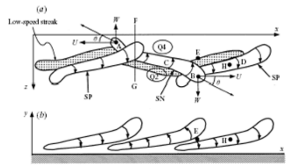
$\kappa = 0.41$, C は流れに依存

4. 剪断乱流における秩序構造の動力学

壁乱流の緩和層
縦渦
ストリーク

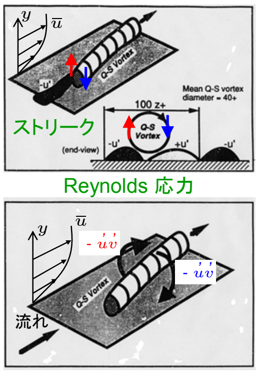


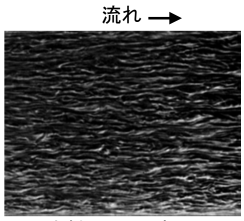
Stretch (1990)



Jeong
Hussain
Schoppa
Kim
(1997)

縦渦による運動量輸送：ストリークの生成

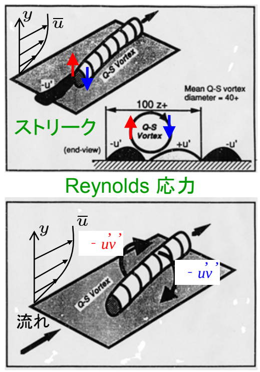


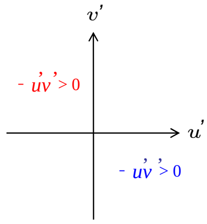


染料による可視化

Robinson (1991)

縦渦による運動量輸送：ストリークの生成



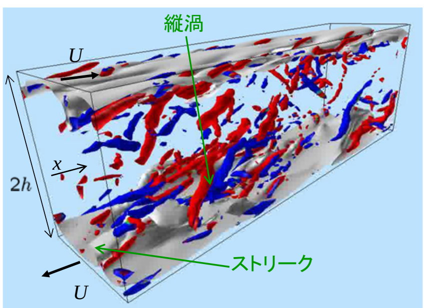


乱流エネルギー生成

$$-\overline{u'v'}\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} > 0$$

Robinson (1991)

縦渦，ストリークの時間発展



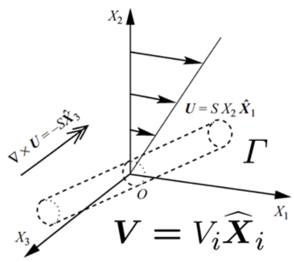
Couette 乱流
$Re \equiv Uh/\nu$
$= 3000$
圧力ラプラシアン
$\nabla^2 p = 3.3\rho(U/h)^2$
赤：$\omega_x > 0$
青：$\omega_x < 0$
主流速度
$u = \pm 0.2U$

K. (2004)

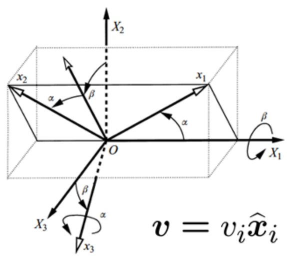
4.1 縦渦の動力学

K., Kida, Tanaka & Yanase (1997)

静止座標系　　構造座標系



$$\boldsymbol{V} = V_i \widehat{\boldsymbol{X}}_i$$



$$\boldsymbol{v} = v_i \widehat{\boldsymbol{x}}_i$$

流れ方向渦
Moore (1985) $\alpha = 0$

$$V_i = M_{i,j} v_j \qquad \widehat{X}_i = M_{i,j} \widehat{x}_j$$

$$\{M_{i,j}\} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha\cos\beta & \cos\alpha\cos\beta & -\sin\beta \\ \sin\alpha\sin\beta & \cos\alpha\sin\beta & \cos\beta \end{pmatrix}$$

$$\boldsymbol{\Omega} = \frac{\mathrm{d}\beta}{\mathrm{d}t}\widehat{\boldsymbol{X}}_1 + \frac{\mathrm{d}\alpha}{\mathrm{d}t}\widehat{\boldsymbol{x}}_3 = \Omega_i \widehat{\boldsymbol{x}}_i$$

$$\Omega_1 = \frac{\mathrm{d}\beta}{\mathrm{d}t}\cos\alpha \quad \Omega_2 = -\frac{\mathrm{d}\beta}{\mathrm{d}t}\sin\alpha \quad \Omega_3 = \frac{\mathrm{d}\alpha}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\widehat{\boldsymbol{x}}_i}{\mathrm{d}t} = \boldsymbol{\Omega} \times \widehat{\boldsymbol{x}}_i$$

$$\frac{\partial \boldsymbol{X}}{\partial t} = \frac{\mathrm{d}x_i}{\mathrm{d}t}\widehat{\boldsymbol{x}}_i + x_i \frac{\mathrm{d}\widehat{\boldsymbol{x}}_i}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \boldsymbol{\Omega} \times \boldsymbol{x} = 0$$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -\boldsymbol{\Omega} \times \boldsymbol{x}$$

$$\begin{aligned} \frac{\partial \boldsymbol{V}}{\partial t} &= \frac{\partial}{\partial t}[v_i(\boldsymbol{x}, t)\widehat{\boldsymbol{x}}_i] \\ &= \frac{\partial v_i}{\partial t}\widehat{\boldsymbol{x}}_i + \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \cdot \nabla\right) v_i \widehat{\boldsymbol{x}}_i + v_i \frac{\mathrm{d}\widehat{\boldsymbol{x}}_i}{\mathrm{d}t} \\ &= \frac{\partial \boldsymbol{v}}{\partial t} - [(\boldsymbol{\Omega} \times \boldsymbol{x}) \cdot \nabla]\boldsymbol{v} + \boldsymbol{\Omega} \times \boldsymbol{v} \end{aligned}$$

Navier-Stokes方程式

$$\frac{\partial \boldsymbol{u}}{\partial t} + [(\boldsymbol{u} - \boldsymbol{\Omega} \times \boldsymbol{x}) \cdot \nabla]\boldsymbol{u} = \boldsymbol{u} \times \boldsymbol{\Omega} - \nabla p + \nu \nabla^2 \boldsymbol{u}$$

渦度方程式

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + [(\boldsymbol{u} - \boldsymbol{\Omega} \times \boldsymbol{x}) \cdot \nabla]\boldsymbol{\omega} = \boldsymbol{\omega} \times \boldsymbol{\Omega} + (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega}$$

一様剪断流と揺らぎ

$$\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}' \quad \boldsymbol{\omega} = \nabla \times \boldsymbol{U} + \boldsymbol{\omega}' \quad p = P + p'$$

$$\boldsymbol{U} = S X_2 \widehat{\boldsymbol{X}}_1 = S M_{1,i} M_{2,j} x_j \widehat{\boldsymbol{x}}_i$$

$$\nabla \times \boldsymbol{U} = -S\widehat{\boldsymbol{X}}_3 = -S M_{3,i} \widehat{\boldsymbol{x}}_i$$

揺らぎの方程式

$$\begin{aligned} &\frac{\partial \boldsymbol{u}'}{\partial t} + [(\boldsymbol{u}' + \overline{\boldsymbol{u}}) \cdot \nabla]\boldsymbol{u}' = \boldsymbol{u}' \times \boldsymbol{\Omega} \\ &\quad -(\boldsymbol{u}' \cdot \nabla)\boldsymbol{U} - \nabla p' + \nu \nabla^2 \boldsymbol{u}' \end{aligned}$$

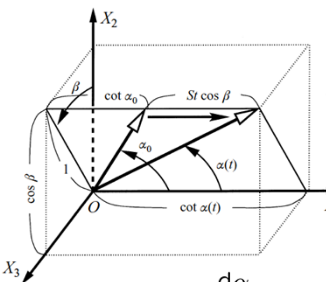
$$\begin{aligned} &\frac{\partial \boldsymbol{\omega}'}{\partial t} + [(\boldsymbol{u}' + \overline{\boldsymbol{u}}) \cdot \nabla]\boldsymbol{\omega}' = \boldsymbol{\omega}' \times \boldsymbol{\Omega} \\ &\quad +(\boldsymbol{\omega}' \cdot \nabla)\boldsymbol{U} + [(\boldsymbol{\omega}' + \nabla \times \boldsymbol{U}) \cdot \nabla]\boldsymbol{u}' + \nu \nabla^2 \boldsymbol{\omega}' \end{aligned}$$

$$\begin{aligned} \overline{\boldsymbol{u}} &= \boldsymbol{U} - \boldsymbol{\Omega} \times \boldsymbol{x} \\ &= S M_{1,i} M_{2,j} x_j \widehat{\boldsymbol{x}}_i - \varepsilon_{i,j,k} \Omega_j x_k \widehat{\boldsymbol{x}}_i \end{aligned}$$

揺らぎが x_1 に依らない $\to$ $\overline{u}_2$, $\overline{u}_3$ が x_1 に依らない

$$\Rightarrow \quad \Omega_2 = 0 \qquad \Omega_3 = -S\sin^2\alpha\cos\beta$$

縦渦の運動



$$\Omega_2 = -\frac{\mathrm{d}\beta}{\mathrm{d}t}\sin\alpha = 0$$

$$\beta = \text{const.}$$

$$\Omega_1 = -\frac{\mathrm{d}\beta}{\mathrm{d}t}\cos\alpha = 0$$

$$\Omega_3 = \frac{\mathrm{d}\alpha}{\mathrm{d}t} = -S\sin^2\alpha\cos\beta$$

$$\cot\alpha = \cot\alpha_0 + S\,t\cos\beta$$

渦軸方向一様揺らぎの基礎方程式

$$\begin{aligned} &\frac{\partial \omega_1'}{\partial t} - \frac{\partial(\psi, \omega_1')}{\partial(x_2, x_3)} - S(\gamma(t)x_2 + \lambda(t)x_3)\frac{\partial \omega_1'}{\partial x_2} \\ &\quad = S\gamma(t)\omega_1' + S\zeta(t)\frac{\partial u_1'}{\partial x_3} + \nu \nabla_\perp^2 \omega_1' \end{aligned}$$

$$\begin{aligned} &\frac{\partial u_1'}{\partial t} - \frac{\partial(\psi, u_1')}{\partial(x_2, x_3)} - S(\gamma(t)x_2 + \lambda(t)x_3)\frac{\partial u_1'}{\partial x_2} \\ &\quad = -S\gamma(t)u_1' - S\left(\cos\alpha\sin\beta\frac{\partial\psi}{\partial x_2} + \cos\beta\frac{\partial\psi}{\partial x_3}\right) \\ &\quad +\nu \nabla_\perp^2 u_1' \end{aligned}$$

$$\gamma(t) = \frac{\partial U_1}{\partial x_1}/S = \cos\alpha\sin\alpha\cos\beta$$

$$\lambda(t) = (\nabla\times\boldsymbol{U})\cdot\hat{\boldsymbol{x}}_1/S = -\sin\alpha\sin\beta$$

$$\zeta(t) = 2\Omega_3/S = -2\sin^2\alpha\cos\beta \quad (\leq 0)$$

$$u_2' = \frac{\partial\psi}{\partial x_3} \qquad u_3' = -\frac{\partial\psi}{\partial x_2} \qquad \nabla_\perp^2\psi = -\omega_1'$$

$$\omega_2' = \frac{\partial u_1'}{\partial x_3} \qquad \omega_3' = -\frac{\partial u_1'}{\partial x_2}$$

Lundgren (1982) 変換

伸長パラメター $A(t)$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = S\gamma(t)A \qquad A(t=0)=1$$

$$A = \exp\left(S\int_0^t \gamma(t')\mathrm{d}t'\right) = \frac{\sin\alpha_0}{\sin\alpha}$$

$$R = [A(t)]^{1/2} r$$

$$S(T-T_0) = S\int_0^t A(t')\mathrm{d}t'$$

$$= \frac{\sin\alpha_0}{2\cos\beta}\Big[\cot\alpha\,\mathrm{cosec}\,\alpha - \cot\alpha_0\,\mathrm{cosec}\,\alpha_0 + \ln\left(\frac{\cot\alpha + \mathrm{cosec}\,\alpha}{\cot\alpha_0 + \mathrm{cosec}\,\alpha_0}\right)\Big]$$

渦軸方向速度の特解

$$\frac{\partial u_1'}{\partial t} - \frac{\partial(\psi, u_1')}{\partial(x_2, x_3)} - S(\gamma(t)x_2 + \lambda(t)x_3)\frac{\partial u_1'}{\partial x_2}$$
$$= -S\gamma(t)u_1' - S\left(\cos\alpha\sin\beta\frac{\partial\psi}{\partial x_2} + \cos\beta\frac{\partial\psi}{\partial x_3}\right)$$
$$+\nu\nabla_\perp^2 u_1'$$

$$u_{1p}' = -S\cos\beta x_2 + S\cos\alpha\sin\beta x_3$$
$$\equiv -Sr\mathrm{Re}[\mathrm{i}f_\infty(t)\mathrm{e}^{-\mathrm{i}\theta}] = -SD(t)r\sin[\varphi(t)-\theta]$$

$$f_\infty(t) = -\cos\alpha\sin\beta - \mathrm{i}\cos\beta = -D(t)\mathrm{e}^{\mathrm{i}\varphi(t)}$$

$$D(t) = (\cos^2\alpha\sin^2\beta + \cos^2\beta)^{\frac{1}{2}}$$

$$\varphi(t) = \arctan\left(\frac{\cos\beta}{\cos\alpha\sin\beta}\right)$$

揺らぎの再定義 $x_2 = r\cos\theta \quad x_3 = r\sin\theta$

$$u_1' = u_{1p}' + u_1''$$

$$\omega(R,\theta,T) = \omega_1'(r,\theta,t)/A(t) = -\nabla_R^2\psi$$

$$Ru(R,\theta,T) = A(t)u_1''(r,\theta,t)$$

変換された基礎方程式

$$-\frac{1}{R}\frac{\partial(\psi,\omega)}{\partial(R,\theta)} + \left(\frac{\partial}{\partial T} - \nu\nabla_R^2\right)\omega$$
$$= S\mathcal{L}_1\omega + S\mathcal{L}_2 u + \frac{2S^2\gamma(t)\lambda(t)}{A(t)^2}$$

$$-\frac{1}{R}\frac{\partial(\psi, Ru)}{\partial(R,\theta)} + \left(\frac{\partial}{\partial T} - \nu\nabla_R^2\right)Ru = S\mathcal{L}_1 Ru$$

$$\mathcal{L}_1 = \frac{1}{2A(t)}\Big[\gamma(t)\left(-\sin\theta\frac{\partial}{\partial\theta} + R\cos 2\theta\frac{\partial}{\partial R}\right)$$
$$+\lambda(t)\left(\cos 2\theta\frac{\partial}{\partial\theta} + R\sin 2\theta\frac{\partial}{\partial R} - \frac{\partial}{\partial\theta}\right)\Big]$$

$$\mathcal{L}_2 = \frac{\zeta(t)}{[A(t)]^{5/2}}\Big[\cos\theta\frac{\partial}{\partial\theta} + \sin\theta\left(R\frac{\partial}{\partial R} + 1\right)\Big]$$

全渦度

$$\omega_1 = S\lambda(t) + A(t)\omega$$

$$\omega_\theta = -[A(t)]^{-1/2}\frac{\partial}{\partial R}(Ru)$$

$$\omega_r = [A(t)]^{-1/2}\frac{\partial u}{\partial\theta}$$

漸近解析：強く細い渦管

$$-\frac{1}{R}\frac{\partial(\psi,\omega)}{\partial(R,\theta)} + \left(\frac{\partial}{\partial T} - \nu\nabla_R^2\right)\omega = 0$$

$$-\frac{1}{R}\frac{\partial(\psi, Ru)}{\partial(R,\theta)} + \left(\frac{\partial}{\partial T} - \nu\nabla_R^2\right)Ru = 0$$

$$\omega = \frac{\Gamma}{4\pi\nu T}\exp\left(-\frac{R^2}{4\nu T}\right)$$

$$u = \mathrm{Re}[\mathrm{i}f(\xi)\mathrm{e}^{-\mathrm{i}\theta}]$$

$$f'' + \left(2\xi + \frac{3}{\xi}\right)f' + \mathrm{i}\frac{\Gamma}{2\pi\nu}\frac{1-\mathrm{e}^{-\xi^2}}{\xi^2}f = 0$$

$$f(\infty) = f_\infty(0)$$

漸近解 Moore (1985)

$$Re = \frac{\Gamma}{2\pi\nu} \gg 1$$

$$\xi \gg (\Gamma/\nu)^{1/4}$$

$$f(\xi) \approx f_\infty(0)\exp\left(\frac{\mathrm{i}Re}{4\xi^2} - \frac{Re^2}{48\xi^6}\right)$$

$$\omega_\theta = -[A(t)]^{-1/2}\mathrm{Re}[\mathrm{i}(\xi f)'\mathrm{e}^{-\mathrm{i}\theta}]$$

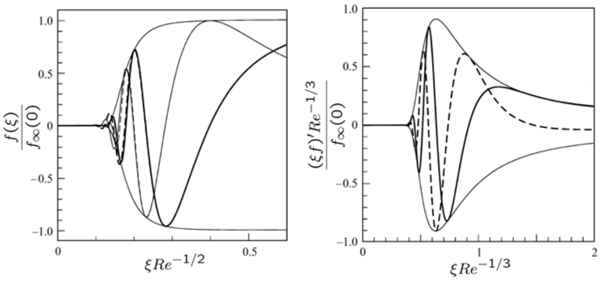
$$(\xi f)' \approx f_\infty(0)\left(1 - \frac{\mathrm{i}Re}{2\xi^2}\right)\exp\left(\frac{\mathrm{i}Re}{4\xi^2} - \frac{Re^2}{48\xi^6}\right)$$

渦管垂直渦度の消滅

$$Re = \frac{\Gamma}{2\pi\nu} = 10^3$$

—— 数値解
------ 漸近解

—— 実数部
------ 虚数部



らせん渦層

$$\frac{\partial u_1''}{\partial t} + \frac{\Gamma}{2\pi r^2}\frac{\partial u_1''}{\partial \theta} = \nu\nabla_\perp^2 u_1''$$

$$u_1''(t=0) = -u_{1p}'(t=0) = SD(0)r\sin[\varphi(0)-\theta]$$

非粘性解

$$u_1'' = SD(0)r\sin\left[\varphi(0) + \frac{\Gamma t}{2\pi r^2} - \theta\right]$$

渦線の渦管垂直断面への射影 $u_1'' = 0$

$$\theta - \frac{\Gamma t}{2\pi r^2} = \mathrm{const.}$$

$$\Delta\theta + \frac{\Gamma t}{\pi r^3}\Delta r = 0$$

らせん間隔 $\Delta\theta \sim 1$

$$\Delta r \sim \frac{r^3}{\Gamma t}$$

らせん渦層の渦度

$$\frac{u_1}{\Delta r} \sim \frac{Sr}{r^3/(\Gamma t)} = \frac{S\Gamma t}{r^2}$$

$$r \to 0 \quad \Longrightarrow \quad \Delta r \to 0 \quad \frac{u_1}{\Delta r} \to \infty$$

粘性解

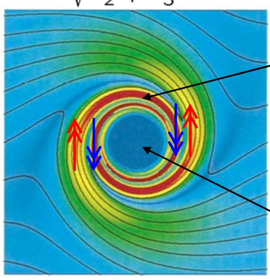
$$u_1'' = SD(0)r\sin\left[\varphi(0) + \frac{\Gamma t}{2\pi r^2} - \theta\right]\exp\left[-\frac{8\pi\nu}{3\Gamma}\left(\frac{\Gamma t}{2\pi r^2}\right)^3\right]$$

粘性の影響

$$\Delta r \sim \frac{r^3}{\Gamma t} \sim \sqrt{\nu t}$$

$\sqrt{\omega_2^2 + \omega_3^2}$

x_2



x_3

$\Gamma/(2\pi\nu) = 10^3$

$(\alpha, \beta) = (\pi/4, 0)$

$r \sim (\Gamma/\nu)^{1/3}\sqrt{\nu t}$

渦管垂直渦度最大

$$\frac{u_1}{\Delta r} \sim S\Gamma t/r^2 \sim S(\Gamma/\nu)^{1/3}$$

$r \ll (\Gamma/\nu)^{1/3}\sqrt{\nu t}$

渦管垂直渦度消滅

一様剪断渦度の伸長

一様剪断渦管垂直渦度の消滅

$$\nabla\times\boldsymbol{U} = S\lambda(t)\hat{\boldsymbol{x}}_1 - S\cos\alpha\sin\beta\hat{\boldsymbol{x}}_2 - S\cos\beta\hat{\boldsymbol{x}}_3$$

$$\nabla\times(u_{1p}'\hat{\boldsymbol{x}}_1) = S\cos\alpha\sin\beta\hat{\boldsymbol{x}}_2 + S\cos\beta\hat{\boldsymbol{x}}_3$$

$$\Longrightarrow \quad \nabla\times\boldsymbol{U} + \nabla\times(u_{1p}'\hat{\boldsymbol{x}}_1) = S\lambda(t)\hat{\boldsymbol{x}}_1$$

$$\frac{\mathrm{d}\omega_{1\,hom}'}{\mathrm{d}t} = S\gamma(t)\omega_{1\,hom}'$$

$$\omega_{1\,hom}'(t) = SA(t)\lambda(0) - S\lambda(t)$$

$$= -S[A(t)\sin\alpha_0 - \sin\alpha]\sin\beta$$

サイクロン $\beta < 0 \Longrightarrow \omega_{1\,hom}' > 0$

近傍場での漸近解析

$$\frac{\Gamma}{r} \gg Sr \implies r \ll (\Gamma/S)^{1/2}$$

$$r \sim (\Gamma/\nu)^{1/3}\sqrt{\nu t} \gg (\Gamma/\nu)^{1/4}\sqrt{\nu t}$$

$$\implies St \ll (\Gamma/\nu)^{1/2}$$

$$\frac{\partial \omega_1'}{\partial t} + \frac{\Gamma}{2\pi r^2}\frac{\partial \omega_1'}{\partial \theta} = S\zeta(t)\frac{\partial u_1'}{\partial x_3} + \nu\nabla_\perp^2 \omega_1'$$

渦軸方向渦度の高次解

$$\frac{\omega_1'}{S^2\zeta(t)} = \frac{4\pi r^2}{\Gamma}\cos\beta$$

$$+D(0)\left\{2t\cos\left[\varphi(0)+\frac{\Gamma t}{2\pi r^2}-\theta\right]\cos\theta\right.$$

$$-\frac{4\pi r^2}{\Gamma}\sin\left[\frac{\Gamma t}{2\pi r^2}+\varphi(0)\right]$$

$$\left.+\frac{2\pi r^2}{\Gamma}\sin\left[\varphi(0)+\frac{\Gamma t}{2\pi r^2}-2\theta\right]\right\}\exp\left[-\frac{8\pi\nu}{3\Gamma}\left(\frac{\Gamma t}{2\pi r^2}\right)^3\right]$$

$$-D(0)\frac{2\pi r^2}{\Gamma}\sin\left[\varphi(0)+\frac{\Gamma t}{\pi r^2}-2\theta\right]\exp\left[-\frac{32\pi\nu}{3\Gamma}\left(\frac{\Gamma t}{2\pi r^2}\right)^3\right]$$

渦度の渦管垂直方向から渦管方向への変換

$$\frac{\omega_1'}{t} \sim S\frac{u_1}{\Delta r} \sim \frac{S^2\Gamma t}{r^2} \implies \omega_1' \sim \frac{S^2\Gamma t^2}{r^2}$$

$$\frac{\Gamma}{r^2}\omega_1' \sim S\frac{u_1}{\Delta r} \sim \frac{S^2\Gamma t}{r^2} \implies \omega_1' \sim S^2 t$$

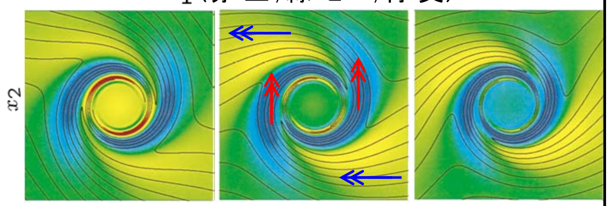
$$r \sim (\Gamma/\nu)^{1/2}\sqrt{\nu t} \qquad \frac{1}{t} \sim \frac{\Gamma}{r^2}$$

$$\omega_1' \sim S^2 t \sim S^2\frac{r^2}{\Gamma}$$

渦管方向渦度揺らぎ

$\Gamma/(2\pi\nu) = 10^3$

ω_1'（赤 正; 緑 ゼロ; 青 負）



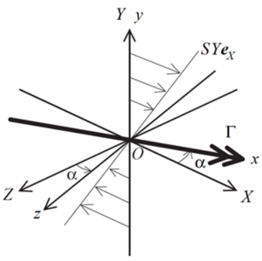
$\alpha = \arctan\sqrt{2}$, $\beta = -\pi/4$ サイクロン

$\alpha = \pi/4$, $\beta = 0$

$\alpha = \arctan\sqrt{2}$, $\beta = +\pi/4$ アンチサイクロン

らせん渦層　K. (2005)

らせん渦層のエネルギー散逸



$$\boldsymbol{u} = u\boldsymbol{e}_x + \frac{\partial\psi}{\partial z}\boldsymbol{e}_y + \left(-\frac{\partial\psi}{\partial y} + Sy\sin\alpha\right)\boldsymbol{e}_z$$

$$\boldsymbol{\omega} = \left(-\nabla^2\psi + S\sin\alpha\right)\boldsymbol{e}_x + \frac{\partial u}{\partial z}\boldsymbol{e}_y - \frac{\partial u}{\partial y}\boldsymbol{e}_z$$

渦度方程式

$$\left(\frac{\partial}{\partial t} - \nu\nabla_\perp^2\right)\nabla_\perp^2\psi - \frac{\partial(\psi,\nabla_\perp^2\psi)}{\partial(y,z)} = -Sy\sin\alpha\frac{\partial}{\partial z}\nabla_\perp^2\psi$$

$$\left(\frac{\partial}{\partial t} - \nu\nabla_\perp^2\right)\nabla_\perp^2\psi - \frac{1}{r}\frac{\partial(\psi,\nabla_\perp^2\psi)}{\partial(r,\theta)}$$

$$= -\frac{1}{2}S\sin\alpha\left(r\sin 2\theta\frac{\partial}{\partial r} + \cos 2\theta\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\theta}\right)\nabla_\perp^2\psi$$

Navier-Stokes方程式

$$\left(\frac{\partial}{\partial t} - \nu\nabla_\perp^2\right)u - \frac{\partial(\psi,u)}{\partial(y,z)} = -Sy\sin\alpha\frac{\partial u}{\partial z}$$

$$\left(\frac{\partial}{\partial t} - \nu\nabla_\perp^2\right)u - \frac{1}{r}\frac{\partial(\psi,u)}{\partial(r,\theta)}$$

$$= -\frac{1}{2}S\sin\alpha\left(r\sin 2\theta\frac{\partial}{\partial r} + \cos 2\theta\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\theta}\right)u$$

エネルギー散逸率

$$\{e_{i,j}\} = \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial u}{\partial y} & \frac{1}{2}\frac{\partial u}{\partial z} \\ \frac{1}{2}\frac{\partial u}{\partial y} & \frac{\partial^2 \psi}{\partial z \partial y} & -\frac{1}{2}\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\psi + \frac{1}{2}S\sin\alpha \\ \frac{1}{2}\frac{\partial u}{\partial z} & -\frac{1}{2}\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\psi + \frac{1}{2}S\sin\alpha & -\frac{\partial^2 \psi}{\partial y \partial z} \end{pmatrix}$$

$$\Phi = 2\nu e_{i,j}e_{i,j} = \nu S^2 + D_T + D_S$$

$$D_T(r,\theta,t) = \nu\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)^2$$

$$+\nu\frac{4}{r^2}\left(\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)^2$$

$$-2\nu S\sin\alpha\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)\cos 2\theta$$

$$+4\nu S\sin\alpha\frac{1}{r}\left(\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)\sin 2\theta$$

$$D_S(r,\theta,t) = \nu\left[\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2 - S^2\cos^2\alpha\right]$$

渦管解 $\alpha = 0$

$$\psi(r,t) = -\frac{\Gamma}{2\pi}\int_0^{\xi}\frac{1-e^{-\xi'^2}}{\xi'}d\xi' \qquad \xi = r/(2\sqrt{\nu t})$$

$$-\nabla^2\psi(r,t) = \frac{\Gamma}{4\pi\nu t}e^{-\xi^2}$$

らせん渦層解 $\alpha = 0$

$$u(r,\theta,t) = Sr\,\mathrm{Re}[f(\xi)\,e^{-i\theta}]$$

$$f'' + \left(2\xi + \frac{3}{\xi}\right)f' + i\frac{\Gamma}{2\pi\nu}\frac{1-e^{-\xi^2}}{\xi^2}f = 0$$

$$f \approx \exp\left(i\frac{Re}{4\xi^2} - \frac{Re^2}{48\xi^6}\right)$$ Moore (1985)

渦管のエネルギー散逸

$$D_T = \nu\left(\frac{\Gamma}{4\pi\nu t}\right)^2\left[\frac{1}{\xi^2} - \left(1 + \frac{1}{\xi^2}\right)e^{-\xi^2}\right]^2$$

らせん渦層のエネルギー散逸

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2 = S^2\left[\frac{1}{2}|\xi f'|^2 + \frac{1}{2}\xi\left(|f|^2\right)' + |f|^2\right]$$

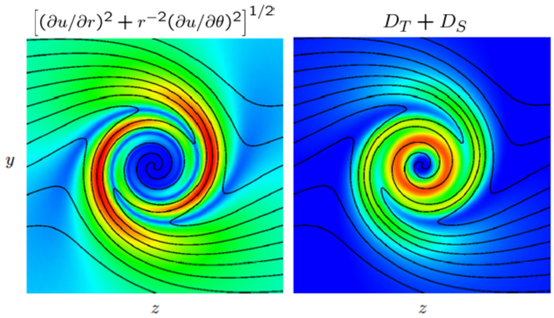
$$+S^2\,\mathrm{Re}[\xi f'\left(f + \tfrac{1}{2}\xi f'\right)e^{-2i\theta}]$$

$$D_S = \nu S^2\left[\frac{1}{2}|\xi f'|^2 + \frac{1}{2}\xi\left(|f|^2\right)' + |f|^2 - 1\right]$$

$$+\nu S^2\,\mathrm{Re}[\xi f'\left(f + \tfrac{1}{2}\xi f'\right)e^{-2i\theta}]$$

渦管まわりのエネルギー散逸

$\alpha = 0 \quad \Gamma/(2\pi\nu) = 100 \quad St = 2.94$



渦管の総エネルギー散逸

$$\int_0^{\infty}\int_0^{2\pi} D_T\, r\,dr\,d\theta = \frac{\Gamma^2}{2\pi t}\int_0^{\infty}\left[\frac{1}{\xi^2} - \left(1 + \frac{1}{\xi^2}\right)e^{-\xi^2}\right]^2\xi d\xi = \frac{\Gamma^2}{8\pi t}$$

らせん渦層の総エネルギー散逸

$$\int_0^{\infty}\int_0^{2\pi} D_S\, r\,dr\,d\theta = 8\pi\nu^2S^2 I_0\, t$$

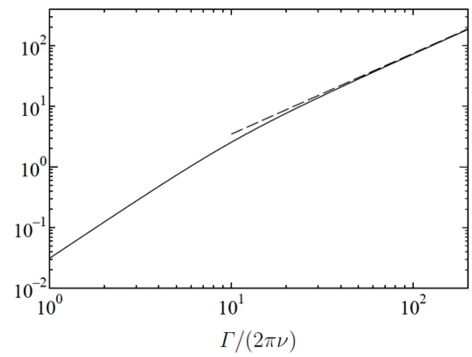
$$I_0 = \int_0^{\infty}\left[\frac{1}{2}|\xi f'|^2 + \frac{1}{2}\xi\left(|f|^2\right)' + |f|^2 - 1\right]\xi d\xi$$

$$I_0 \approx 2^{-3}\left(\frac{\Gamma}{2\pi\nu}\right)^{4/3}\int_0^{\infty}\frac{1}{\xi^3}\exp\left(-\frac{1}{24\xi^6}\right)d\xi$$

$$= 2^{-3}3^{-2/3}\Gamma(\tfrac{1}{3})\left(\frac{\Gamma}{2\pi\nu}\right)^{4/3}$$

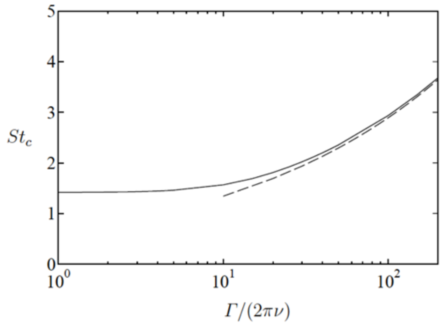
$$\int_0^{\infty}\int_0^{2\pi} D_S\, r\,dr\,d\theta \approx 1.29\pi\nu^2S^2\left(\frac{\Gamma}{2\pi\nu}\right)^{4/3} t$$

積分の漸近評価



臨界時間　渦管の散逸＝らせん渦層の散逸

$$St_c = \tfrac{1}{4} I_0^{-1/2} \frac{\Gamma}{2\pi\nu} \qquad St_c \approx 0.623 \left(\frac{\Gamma}{2\pi\nu}\right)^{1/3}$$



渦管のエネルギー散逸

速度 $\dfrac{\Gamma}{\sqrt{\nu t}}$　変形速度 $\dfrac{\Gamma}{\nu t}$ $\Rightarrow$ $\nu \left(\dfrac{\Gamma}{\nu t}\right)^2$

$$\nu \left(\frac{\Gamma}{\nu t}\right)^2 \nu t \sim \frac{\Gamma^2}{t}$$

らせん渦層のエネルギー散逸

速度 Sr　変形速度 $\dfrac{Sr}{\Delta r} \sim S\dfrac{\Gamma t}{r^2}$

$$r \sim \left(\frac{\Gamma}{\nu}\right)^{1/3} \sqrt{\nu t} \Rightarrow \nu \left(S\frac{\Gamma t}{r^2}\right)^2 \sim \nu S^2 \left(\frac{\Gamma}{\nu}\right)^{2/3}$$

$$\nu S^2 \left(\frac{\Gamma}{\nu}\right)^{2/3} \left(\frac{\Gamma}{\nu}\right)^{2/3} \nu t \sim \nu^2 S^2 \left(\frac{\Gamma}{\nu}\right)^{4/3} t$$

臨界時間

$$\frac{\Gamma^2}{t} \sim \nu^2 S^2 \left(\frac{\Gamma}{\nu}\right)^{4/3} t \Rightarrow St_c \sim \left(\frac{\Gamma}{\nu}\right)^{1/3}$$

Burgers渦管　対　らせん渦層

速度 $\dfrac{\Gamma}{\sqrt{\nu/\gamma}}$ 変形速度 $\dfrac{\Gamma}{\nu/\gamma}$ $\Rightarrow$ $\nu \left(\dfrac{\Gamma}{\nu/\gamma}\right)^2$

$$\nu \left(\frac{\Gamma}{\nu/\gamma}\right)^2 \nu/\gamma \sim \Gamma^2 \gamma$$

$$\Rightarrow St_c \sim \left(\frac{\Gamma}{\nu}\right)^{2/3} \frac{\gamma}{S}$$

漸近解析：渦管垂直方向剪断の影響

$$St|\alpha| \ll 1$$

$$\psi(r,\theta,t) = \Gamma\left[\psi_0(\xi) + St\alpha\,\psi_1(\xi,\theta) + \cdots\right]$$

$$u(r,\theta,t) = Sr\left[u_0(\xi,\theta) + St\alpha\,u_1(\xi,\theta) + \cdots\right]$$

$$\sin\alpha = \alpha - \tfrac{1}{6}\alpha^3 + \cdots$$

高次流れ関数（渦管）

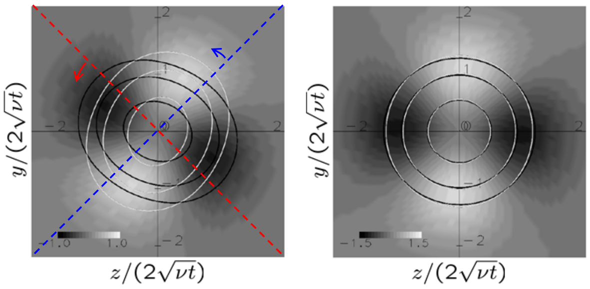
$$\psi_1 = \mathrm{Re}\left[g(\xi)\,\mathrm{e}^{-2\mathrm{i}\theta}\right] \qquad tg(\xi=\infty) = 0$$

$$g^{\mathrm{iv}} + 2\left(\xi + \frac{1}{\xi}\right) g''' + \left(2 - \frac{9}{\xi^2}\right) g'' - \frac{1}{\xi}\left(10 - \frac{9}{\xi^2}\right) g' + \frac{16}{\xi^2} g$$
$$+ \frac{\mathrm{i}}{\pi}\frac{\Gamma}{\nu}\left[\frac{1-\mathrm{e}^{-\xi^2}}{\xi^2}\left(g'' + \frac{1}{\xi}g' - \frac{4}{\xi^2}g\right) + 4\,\mathrm{e}^{-\xi^2} g\right] = \frac{4\mathrm{i}}{\pi}\xi^2 \mathrm{e}^{-\xi^2}$$

高次渦管方向渦度

$\alpha = \pm 0.5$，$St = 1$

$\Gamma/(2\pi\nu) = 10$　　$\Gamma/(2\pi\nu) = 100$



$$-\nabla_\perp^2 \psi = \tfrac{1}{2}\mathrm{e}^{-1}/(4\pi),\ \mathrm{e}^{-1}/(4\pi),\ 2\mathrm{e}^{-1}/(4\pi)$$

高次速度（らせん渦層）

$$u_1 = \mathrm{Re}\left\{\mathrm{i}\left[f_1(\xi)\,\mathrm{e}^{-\mathrm{i}\theta} + f_3(\xi)\,\mathrm{e}^{-3\mathrm{i}\theta}\right]\right\}$$

$$rtf_1(\xi=\infty) = 0,\ rtf_3(\xi=\infty) = 0$$

$$f_1'' + \left(2\xi + \frac{3}{\xi}\right) f_1' - 4f_1 + \mathrm{i}\frac{\Gamma}{2\pi\nu}\frac{1-\mathrm{e}^{-\xi^2}}{\xi^2} f_1 = \zeta(\xi)$$

$$f_3'' + \left(2\xi + \frac{3}{\xi}\right) f_3' - 4\left(1 + \frac{2}{\xi^2}\right) f_3 + 3\mathrm{i}\frac{\Gamma}{2\pi\nu}\frac{1-\mathrm{e}^{-\xi^2}}{\xi^2} f_3 = \chi(\xi)$$

$$\zeta = -\frac{\Gamma}{\nu}\frac{1}{\xi^2}\left(\bar{f}g + \xi\bar{f}'g + \tfrac{1}{2}\xi\bar{f}g'\right) + \xi\bar{f}' - 4\mathrm{i}\,\mathrm{Im}(f)$$

$$\chi = -\frac{\Gamma}{\nu}\frac{1}{\xi^2}\left(fg + \xi f'g - \tfrac{1}{2}\xi fg'\right) + \xi f'$$

$\overline{(\,)}$ 複素共役

高Reynolds数渦管の遠方場 $\xi \gg (\Gamma/\nu)^{1/4}$

$$f_1'' + \left(2\xi + \frac{3}{\xi}\right) f_1' - \left(4f_1 - \mathrm{i}\frac{\Gamma}{2\pi\nu}\frac{1}{\xi^2}\right) f_1 = \zeta$$

$$f_3'' + \left(2\xi + \frac{3}{\xi}\right) f_3' - \left[4 + \left(8 - 3\mathrm{i}\frac{\Gamma}{2\pi\nu}\right)\frac{1}{\xi^2}\right] f_3 = \chi$$

$$\zeta \approx \xi\bar{f}' - 4\mathrm{i}\,\mathrm{Im}(f)$$
$$\approx \mathrm{i}\frac{Re}{2\xi^2}\exp\left(-\mathrm{i}\frac{Re}{4\xi^2} - \frac{Re^2}{48\xi^6}\right) - 4\mathrm{i}\sin\left(\frac{Re}{4\xi^2}\right)\exp\left(-\frac{Re^2}{48\xi^6}\right)$$

$$\chi \approx \xi f' \approx -\mathrm{i}\frac{Re}{2\xi^2}\exp\left(\mathrm{i}\frac{Re}{4\xi^2} - \frac{Re^2}{48\xi^6}\right)$$

漸近解（高次らせん渦層）

θ 依存解

$$\mathrm{Re}(f_1) = \tfrac{3}{4}\left[\cos\left(\frac{Re}{4\xi^2}\right) - \frac{4\xi^2}{Re}\sin\left(\frac{Re}{4\xi^2}\right)\right]\exp\left(-\frac{Re^2}{48\xi^6}\right)$$

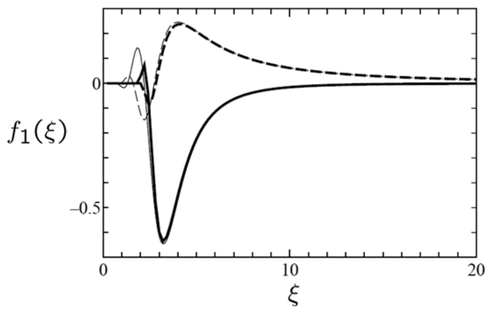
$$\mathrm{Im}(f_1) = \tfrac{1}{4}\sin\left(\frac{Re}{4\xi^2}\right)\exp\left(-\frac{Re^2}{48\xi^6}\right)$$

3θ 依存解

$$f_3 = -\left(1 - \frac{\mathrm{i}\xi^2}{2Re}\right)\exp\left(\mathrm{i}\frac{Re}{4\xi^2} - \frac{Re^2}{48\xi^6}\right) - \frac{\mathrm{i}\xi^2}{2Re}\exp\left(\mathrm{i}\frac{3Re}{4\xi^2} - \frac{3Re^2}{16\xi^6}\right)$$

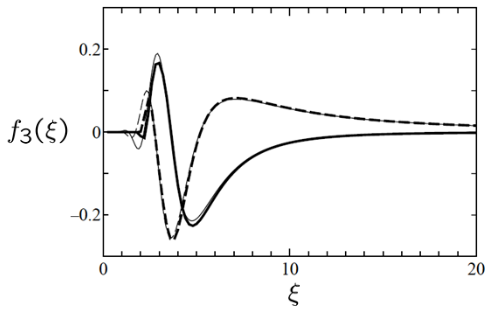
漸近解と数値解

$\Gamma/(2\pi\nu) = 100$



漸近解と数値解

$\Gamma/(2\pi\nu) = 100$



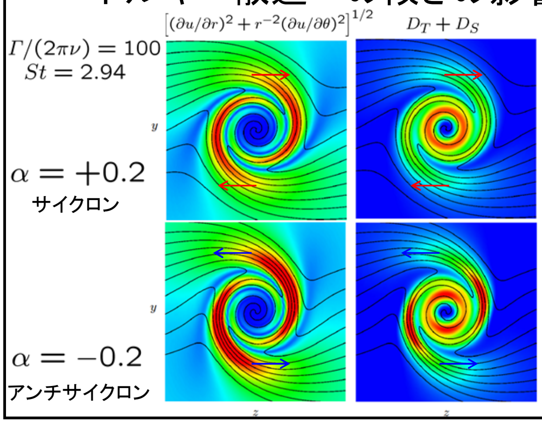
渦管のエネルギー散逸

$$D_T = \nu\left(\frac{\Gamma}{4\pi\nu t}\right)^2\left[\frac{1}{\xi^2} - \left(1 + \frac{1}{\xi^2}\right)\mathrm{e}^{-\xi^2}\right]^2$$
$$+\nu\frac{St\alpha}{8\pi}\left(\frac{\Gamma}{\nu t}\right)^2\left[\frac{1}{\xi^2} - \left(1 + \frac{1}{\xi^2}\right)\mathrm{e}^{-\xi^2}\right]\mathrm{Re}\left[\left(g'' - \frac{1}{\xi}g' + \frac{4}{\xi^2}g\right)\mathrm{e}^{-2\mathrm{i}\theta}\right]$$
$$-\nu\frac{S\alpha}{2\pi}\frac{\Gamma}{\nu t}\left[\frac{1}{\xi^2} - \left(1 + \frac{1}{\xi^2}\right)\mathrm{e}^{-\xi^2}\right]\cos 2\theta$$

らせん渦層のエネルギー散逸

$$D_S = \nu S^2\left[\tfrac{1}{2}|\xi f'|^2 + \tfrac{1}{2}\xi\left(|f|^2\right)' + |f|^2 - 1\right]$$
$$+\nu S^2\,\mathrm{Re}[\xi f'\left(f + \tfrac{1}{2}\xi f'\right)\mathrm{e}^{-2\mathrm{i}\theta}]$$
$$-2\nu S^3 t\alpha\mathrm{Re}\left\{\mathrm{i}\left[\tfrac{1}{2}\xi^2 f'\bar{f}_1' + \tfrac{1}{2}\xi\left(f\bar{f}_1\right)' + f\bar{f}_1\right]\right\}$$
$$+2\nu S^3 t\alpha\mathrm{Re}\left\{\mathrm{i}\left[\tfrac{1}{2}\xi^2\left(\bar{f}' f_3' + f' f_1'\right) + \tfrac{1}{2}\xi\left(\bar{f} f_3 + f f_1\right)' + 2\bar{f} f_3\right]\mathrm{e}^{-2\mathrm{i}\theta}\right\}$$
$$+2\nu S^3 t\alpha\mathrm{Re}\{\mathrm{i}[\tfrac{1}{2}\xi^2 f' f_3' + \tfrac{1}{2}\xi(f f_3)' - f f_3]\mathrm{e}^{-4\mathrm{i}\theta}\}$$

エネルギー散逸への傾きの影響



らせん渦層の総エネルギー散逸への傾きの影響

$$\int_0^\infty \int_0^{2\pi} D_S\, r\mathrm{d}r\mathrm{d}\theta = 8\pi\nu^2 S^2 I_0\, t - 16\pi\nu^2 S^3 \alpha I_1\, t^2$$

$$I_1 = \int_0^\infty \mathrm{Re}\left\{\mathrm{i}\left[\tfrac{1}{2}\xi^2 f'\bar{f}_1' + \tfrac{1}{2}\xi\left(f\bar{f}_1\right)' + f\bar{f}_1\right]\right\}\xi\mathrm{d}\xi$$

$$I_1 \approx \frac{Re}{32} I_2 + \frac{Re}{16}$$

$$I_2 = 3\int_0^\infty \left[-\frac{2}{\xi^2}\sin\xi + \frac{1}{\xi}(1+\cos\xi)\right]\exp\left(-\frac{\xi^3}{3Re}\right)\mathrm{d}\xi$$

$$\frac{\mathrm{d}I_2}{\mathrm{d}Re} = \frac{1}{Re} + \frac{1}{Re^2}\int_0^\infty \left(-2\xi\sin\xi + \xi^2\cos\xi\right)\exp\left(-\frac{\xi^3}{3Re}\right)\mathrm{d}\xi$$

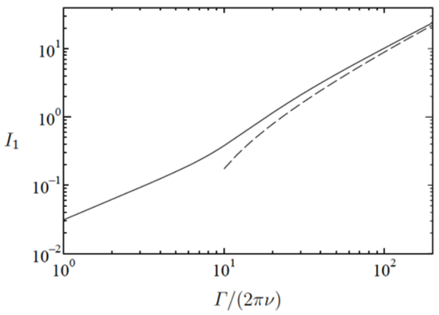
$$\frac{\mathrm{d}I_2}{\mathrm{d}Re} \approx \frac{1}{Re}$$

$$I_2 \approx \ln Re + C \quad (C = -3.75)$$

$$I_1 \approx \frac{Re}{32}\ln Re + \frac{C+2}{32}Re \quad \left(\frac{C+2}{32} = -0.0546\right)$$

積分の漸近評価

$$-16\pi\nu^2 S^3 \alpha I_1\, t^2 \approx -\tfrac{1}{4}\alpha\nu S^3 \Gamma \ln\left(\frac{\Gamma}{2\pi\nu}\right) t^2$$

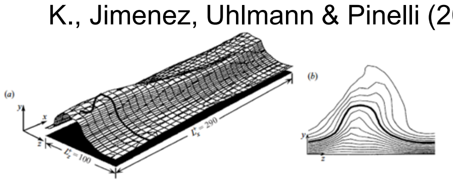


縦渦の生成機構

- 縦渦による再生（非粘性）
  Sendstad & Moin (1992)
- 縦渦による再生（壁面）
  Brooke & Hanratty (1993)
- 外層流による生成
  Sreenivasan (1988)
- ストリークの不安定化による再生
  Jimenez & Moin (1991); Hamilton, Kim & Waleffe (1995); Waleffe (1997); Jimenez & Pinelli (1999); Schoppa & Hussain (2002)

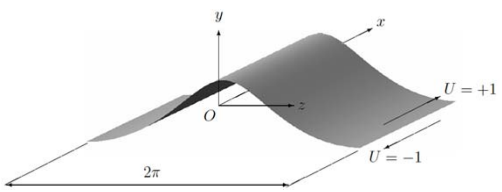
4.2 ストリークの不安定（縦渦の生成）

K., Jimenez, Uhlmann & Pinelli (2003)



Schoppa
Hussain
(2002)

ストリークの不安定⇒波状渦層の不安定



2次元渦層の不安定

渦層（厚さ無限小）

$$\boldsymbol{u} = U\boldsymbol{e}_x + \nabla\phi \qquad \nabla^2\phi = 0$$

Bernoulli方程式

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi + p = 0$$

渦層（実質面） $S(y,z) + s(x,y,z,t) = 0$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)s + \frac{\partial\phi}{\partial y}\frac{\partial S}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial S}{\partial z} = 0$$

ノーマル・モード $f=\phi,\ p,\ s$

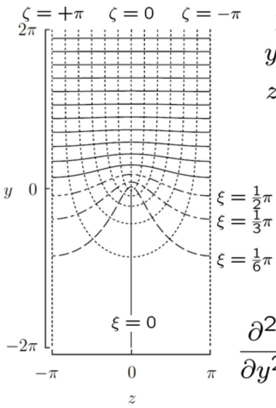
$$f(x,y,z,t)=\mathrm{Re}\left\{\tilde{f}(y,z)\,\mathrm{e}^{\mathrm{i}\alpha(x-ct)}\right\}$$

$$\frac{\partial^2\tilde{\phi}}{\partial y^2}+\frac{\partial^2\tilde{\phi}}{\partial z^2}-\alpha^2\tilde{\phi}=0$$

$$\mathrm{i}\alpha(U-c)\tilde{\phi}+\tilde{p}=0$$

$$\mathrm{i}\alpha(U-c)\tilde{s}+\left(\frac{\partial\tilde{\phi}}{\partial y}\frac{\partial}{\partial y}+\frac{\partial\tilde{\phi}}{\partial z}\frac{\partial}{\partial z}\right)S=0$$

等角写像: 自由渦層



$$\chi=y+\mathrm{i}z\leftrightarrow\sigma=\zeta+\mathrm{i}\xi$$

$$y\in(-\infty,\infty)\quad \xi\in(0,\infty)$$

$$z\in(-\pi,\pi)\quad \zeta\in(-\pi,\pi)$$

$$\chi=2\log\left(\cos\tfrac{1}{2}\sigma\right)$$

$$\sigma=2\,\mathrm{arccos}\left(\mathrm{e}^{\frac{1}{2}\chi}\right)$$

$$S(y,z)\equiv\xi-\xi_0=0$$

$$\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=J\left(\frac{\partial^2}{\partial\zeta^2}+\frac{\partial^2}{\partial\xi^2}\right)$$

ヤコビ行列式

$$\chi=y+\mathrm{i}z\leftrightarrow\sigma=\zeta+\mathrm{i}\xi$$

$$J\equiv\frac{\partial(\zeta,\xi)}{\partial(y,z)}=\left|\cot\tfrac{1}{2}\sigma\right|^2=\frac{\cosh\xi+\cos\zeta}{\cosh\xi-\cos\zeta}$$

$$J^{-1}=\sum_{m=0}^{\infty}j_m(\xi)\cos m\zeta$$

$$j_0=1+2\mathrm{e}^{-\xi}/\sinh\xi$$

$$j_m=4(-1)^m\mathrm{e}^{-m\xi}/\tanh\xi\ \ (m\geq 1)$$

写像空間での撹乱方程式

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\zeta^2}\right)\tilde{\phi}-\alpha^2J^{-1}\tilde{\phi}=0$$

$$\mathrm{i}\alpha(U-c)\tilde{\phi}+\tilde{p}=0$$

$$-\mathrm{i}\alpha(U-c)\tilde{\xi}+J\frac{\partial\tilde{\phi}}{\partial\xi}=0\quad(\xi=\xi_0)$$

$$\tilde{\xi}(\zeta)=-\tilde{s}(y,z)$$

$$\xi=\xi_0+\mathrm{Re}\left\{\tilde{\xi}(\zeta)\,\mathrm{e}^{\mathrm{i}\alpha(x-ct)}\right\}$$

接合条件 $(\xi=\xi_0)$

$$\tilde{\phi}(\xi,\zeta)=\begin{cases}\tilde{\phi}^+(\xi,\zeta) & (\xi>\xi_0)\\ \tilde{\phi}^-(\xi,\zeta) & (\xi<\xi_0)\end{cases}$$

$$(1-c)\tilde{\phi}^+=(-1-c)\tilde{\phi}^-$$

$$(-1-c)\frac{\partial\tilde{\phi}^+}{\partial\xi}=(1-c)\frac{\partial\tilde{\phi}^-}{\partial\xi}$$

長波長近似 $(\alpha\ll 1)$

$$\tilde{\phi}^{\pm}=\alpha\left\{\tilde{\phi}_0^{\pm}(\xi,\zeta)+\alpha^2\tilde{\phi}_2^{\pm}(\xi,\zeta)+\cdots\right\}$$

$$c=c_0+\alpha^2c_2+\cdots$$

$$\tilde{\xi}=\tilde{\xi}_0(\zeta)+\alpha^2\tilde{\xi}_2(\zeta)+\cdots$$

主要オーダー

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\zeta^2}\right)\tilde{\phi}_0^{\pm}=0$$

$$(1-c_0)\tilde{\phi}_0^+ +(1+c_0)\tilde{\phi}_0^-=0\quad(\xi=\xi_0)$$

$$(1+c_0)\frac{\partial\tilde{\phi}_0^+}{\partial\xi}+(1-c_0)\frac{\partial\tilde{\phi}_0^-}{\partial\xi}=0\quad(\xi=\xi_0)$$

高次オーダー

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\zeta^2}\right)\tilde\phi_2^\pm = J^{-1}\tilde\phi_0^\pm$$

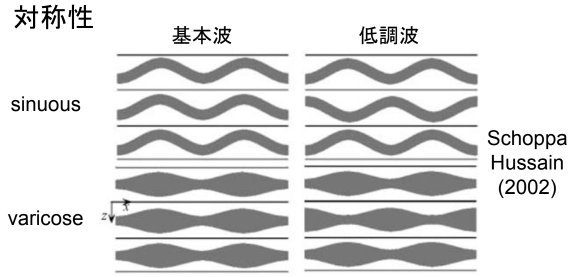
$$(1-c_0)\tilde\phi_2^+ + (1+c_0)\tilde\phi_2^- = c_2\left[\tilde\phi_0\right] \quad (\xi=\xi_0)$$

$$(1+c_0)\frac{\partial\tilde\phi_2^+}{\partial\xi}+(1-c_0)\frac{\partial\tilde\phi_2^-}{\partial\xi} = -c_2\left[\frac{\partial\tilde\phi_0}{\partial\xi}\right] \quad (\xi=\xi_0)$$

$$\left[\tilde\phi_0\right]=\tilde\phi_0^+|_{\xi=\xi_0}-\tilde\phi_0^-|_{\xi=\xi_0}$$

$$\left[\frac{\partial\tilde\phi_0}{\partial\xi}\right] = \left.\frac{\partial\tilde\phi_0^+}{\partial\xi}\right|_{\xi=\xi_0} - \left.\frac{\partial\tilde\phi_0^-}{\partial\xi}\right|_{\xi=\xi_0}$$

対称性



主要オーダー解（sinuous）

$$\tilde\phi_0^\pm = \tilde\phi_{0,n}^\pm(\xi)\sin n\zeta \qquad (n=1,2,3,\cdots)$$

$$\left(\frac{d^2}{d\xi^2}-n^2\right)\tilde\phi_{0,n}^\pm=0$$

$$\tilde\phi^-_{0,n}(\xi=0)=0 \qquad \tilde\phi^+_{0,n}(\xi=\infty)=0$$

$$\tilde\phi^-_{0,n}=A_-\sinh n\xi \qquad \tilde\phi^+_{0,n}=B_+e^{-n\xi}$$

$$(1-c_0)\,e^{-n\xi_0}B_+ + (1+c_0)\sinh n\xi_0\,A_- = 0$$

$$-(1+c_0)\,e^{-n\xi_0}B_+ + (1-c_0)\cosh n\xi_0\,A_-=0$$

$$c_0 = e^{-2n\xi_0}\pm i\left(1-e^{-4n\xi_0}\right)^{1/2}$$

$$\tilde\phi^-_{0,n}\propto(1-c_0)\,e^{-n\xi_0}\sinh n\xi$$

$$\tilde\phi^+_{0,n}\propto -(1+c_0)\,e^{-n\xi}\sinh n\xi_0$$

主要オーダー解（varicose）

$$\tilde\phi_0=\begin{cases}\tilde\phi^+_{0,n}(\xi)\cos n\zeta+(-1)^n\dfrac{1+c_0}{1-c_0}\tilde\phi^-_{0,n}(0) & (\xi>\xi_0)\\ \tilde\phi^-_{0,n}(\xi)\cos n\zeta-(-1)^n\tilde\phi^-_{0,n}(0) & (\xi<\xi_0)\end{cases}$$

$$\left.\frac{d\tilde\phi^-_{0,n}}{d\xi}\right|_{\xi=0}=0 \qquad \tilde\phi^+_{0,n}(\xi=\infty)=0$$

$$\tilde\phi^-_{0,n}=A_-\cosh n\xi \qquad \tilde\phi^+_{0,n}=B_+e^{-n\xi}$$

$$c_0=-e^{-2n\xi_0}\pm i\left(1-e^{-4n\xi_0}\right)^{1/2}$$

$$\tilde\phi^-_{0,n}\propto(1-c_0)\,e^{-n\xi_0}\cosh n\xi$$

$$\tilde\phi^+_{0,n}\propto-(1+c_0)\,e^{-n\xi}\cosh n\xi_0$$

固有関数の構造

$n\xi_0\gg1 \quad \Rightarrow \quad c_0=\pm i,\quad \tilde\phi^-_{0,n}=\pm i\,e^{n(\xi-\xi_0)},\quad \tilde\phi^+_{0,n}=e^{-n(\xi-\xi_0)}$

一対の斜め波 $(\alpha,\pm n)$

Kelvin-Helmholtz不安定

↑

$\xi_0\gg1 \quad \Rightarrow$ 平面状渦層の不安定

↑

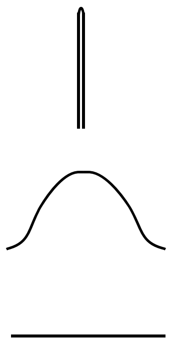
$n\gg1 \quad \Rightarrow$ 波数 >> 渦面曲率

増幅率 $\alpha\mathrm{Im}(c_0)$

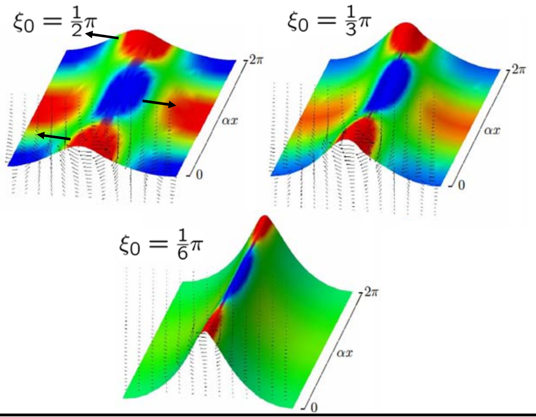
$\xi_0=0 \qquad \alpha\mathrm{Im}(c_0)=0$

↓ 波打ち小　↓ 増幅率大

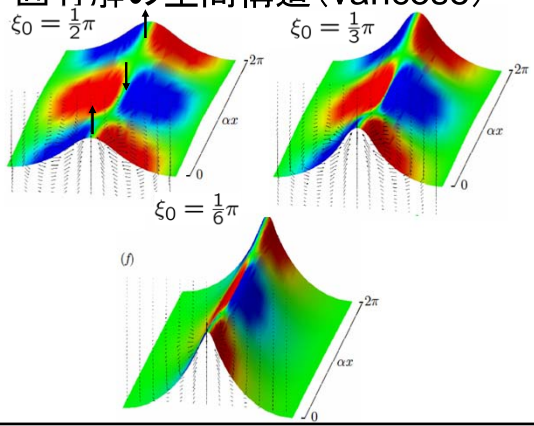
$\xi_0=\infty \qquad \alpha\mathrm{Im}(c_0)=\alpha$

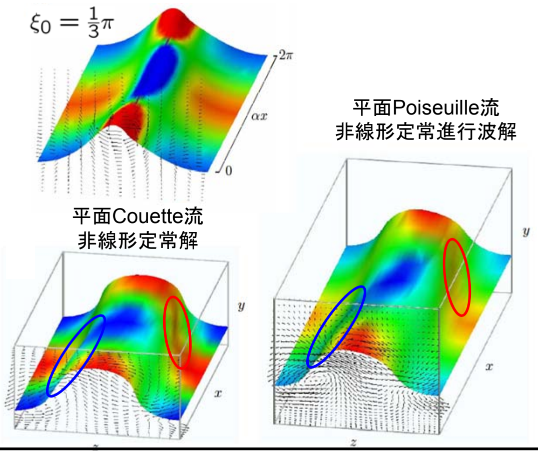


固有解の空間構造（sinuous）



固有解の空間構造（varicose）





非線形効果：渦度伸長

Schoppa & Hussain (2002)

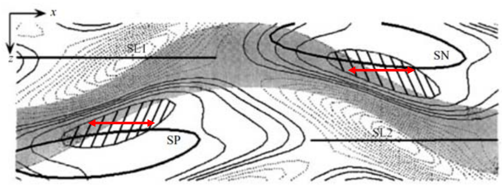


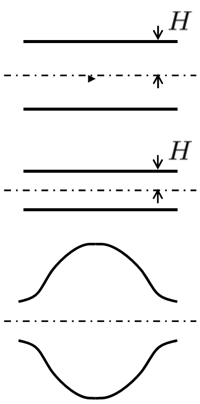
FIGURE 29. Generation of positive (solid) and negative (dotted) $\partial u/\partial x$ by STG-induced waviness of a streak (shaded) at $y^+ = 20$. The cross-section through the positive (SP) and negative (SN) streamwise vortices are denoted by solid bold contours of $|\omega_x|$. The hatched regions denote the largest values of vortex stretching by $+\partial u/\partial x$, while lines SL1 and SL2 identify internal shear layers containing $-\partial u/\partial x$.

非貫通壁面の影響

増幅率大

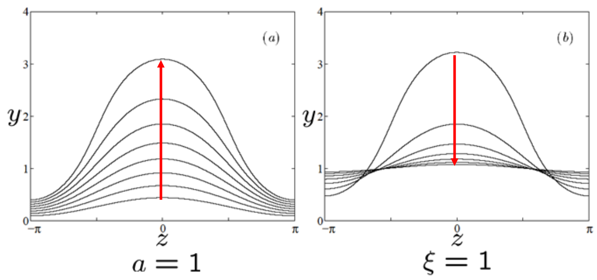
$$\alpha \mathrm{Im}(c_0) = \alpha \left(1 - \mathrm{e}^{-4nH}\right)^{1/2}$$

増幅率？



等角写像：非貫通壁面上の渦層

$$\chi = \log \sin\left(\frac{\mathrm{i}a + \sigma}{2}\right) - \log \sin\left(\frac{\mathrm{i}a - \sigma}{2}\right)$$



$a = 1$

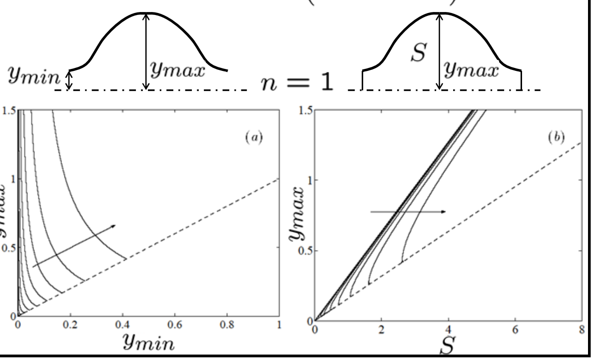
$\xi/a = 0.2(0.1)0.9$

$\xi = 1$

$a/\xi = 1.1(0.4)3.5$

壁面上の渦層の不安定性

$$c_0 = -\mathrm{e}^{-2n\xi_0} \pm \mathrm{i}\left(1 - \mathrm{e}^{-4n\xi_0}\right)^{1/2}$$



高次オーダー解（sinuous）

$$J^{-1}\tilde{\phi}_0^{\pm} = \sum_{m=1}^{\infty} g_m^{\pm} \sin m\zeta$$

$$g_m^{\pm}(\xi) = \begin{cases} \frac{1}{2}(2j_0 - j_{2n})\tilde{\phi}_{0,n}^{\pm} & (m = n) \\ \frac{1}{2}(j_{|m-n|} - j_{m+n})\tilde{\phi}_{0,n}^{\pm} & (m \neq n) \end{cases}$$

$$\tilde{\phi}_2^{\pm} = \sum_{m=1}^{\infty} \tilde{\phi}_{2,m}^{\pm}(\xi) \sin m\zeta$$

高次オーダー解（varicose）

$$J^{-1}\tilde{\phi}_0^{\pm} = \sum_{m=0}^{\infty} g_m^{\pm} \cos m\zeta$$

$$g_m^{\pm}(\xi) = \begin{cases} \frac{1}{2}j_n\tilde{\phi}_{0,n}^{\pm} + j_0 G^{\pm} & (m = 0) \\ \frac{1}{2}(2j_0 + j_{2n})\tilde{\phi}_{0,n}^{\pm} + j_n G^{\pm} & (m > 0,\ m = n) \\ \frac{1}{2}(j_{|m-n|} + j_{m+n})\tilde{\phi}_{0,n}^{\pm} + j_m G^{\pm} & (m > 0,\ m \neq n) \end{cases}$$

$$G^{-} = -(-1)^n \tilde{\phi}_{0,n}^{-}(0)$$

$$G^{+} = (-1)^n \frac{1 + c_0}{1 - c_0} \tilde{\phi}_{0,n}^{-}(0)$$

$$\tilde{\phi}_2^{\pm} = \sum_{m=0}^{\infty} \tilde{\phi}_{2,m}^{\pm}(\xi) \cos m\zeta$$

高次方程式

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - m^2\right)\tilde{\phi}_{2,m}^{\pm} = g_m^{\pm}$$

$$(1 - c_0)\tilde{\phi}_{2,m}^{+} + (1 + c_0)\tilde{\phi}_{2,m}^{-} = c_2\,\delta_{mn}\left[\tilde{\phi}_{0,n}\right]$$

$$(1 + c_0)\frac{\mathrm{d}\tilde{\phi}_{2,m}^{+}}{\mathrm{d}\xi} + (1 - c_0)\frac{\mathrm{d}\tilde{\phi}_{2,m}^{-}}{\mathrm{d}\xi} = -c_2\,\delta_{mn}\left[\frac{\mathrm{d}\tilde{\phi}_{0,n}}{\mathrm{d}\xi}\right]$$

可解条件

$$\int_0^{\infty} \tilde{\phi}_{0,m}\left\{\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - m^2\right)\tilde{\phi}_{2,m}\right\}\mathrm{d}\xi = \int_0^{\infty} \tilde{\phi}_{0,m}\, g_m\, \mathrm{d}\xi$$

$$\int_0^{\infty} \left\{\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - m^2\right)\tilde{\phi}_{0,m}\right\}\tilde{\phi}_{2,m}\,\mathrm{d}\xi$$

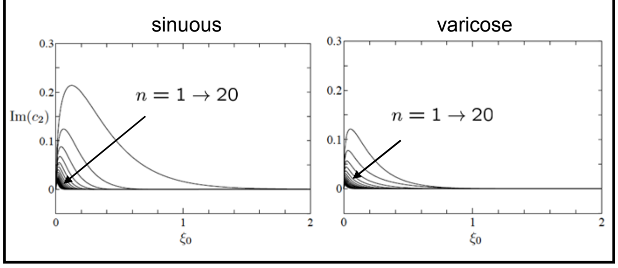
$$+\left[\tilde{\phi}_{2,m}\frac{\mathrm{d}\tilde{\phi}_{0,m}}{\mathrm{d}\xi} - \frac{\mathrm{d}\tilde{\phi}_{2,m}}{\mathrm{d}\xi}\tilde{\phi}_{0,m}\right] = \int_0^{\infty} g_m\tilde{\phi}_{0,m}\mathrm{d}\xi$$

$$\left[\tilde{\phi}_{2,m}\frac{\mathrm{d}\tilde{\phi}_{0,m}}{\mathrm{d}\xi} - \frac{\mathrm{d}\tilde{\phi}_{2,m}}{\mathrm{d}\xi}\tilde{\phi}_{0,m}\right] = \int_0^{\infty} g_m\tilde{\phi}_{0,m}\mathrm{d}\xi$$

$$\left[\tilde{\phi}_{2,m}\frac{\mathrm{d}\tilde{\phi}_{0,m}}{\mathrm{d}\xi} - \frac{\mathrm{d}\tilde{\phi}_{2,m}}{\mathrm{d}\xi}\tilde{\phi}_{0,m}\right] = \frac{4c_2}{1 - c_0^2}\,\tilde{\phi}_{0,n}^{+}\frac{\mathrm{d}\tilde{\phi}_{0,n}^{+}}{\mathrm{d}\xi}\bigg|_{\xi=\xi_0}$$

高次オーダー固有値（自由渦層）

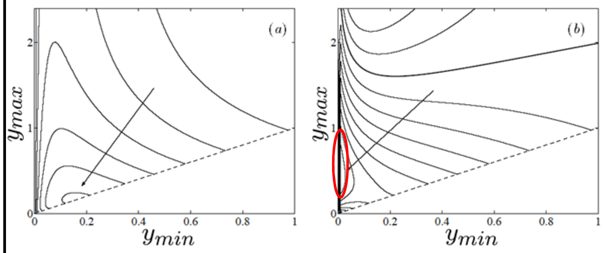
$$c_2 = \frac{\int_0^{\infty} g_n\tilde{\phi}_{0,n}\mathrm{d}\xi}{\frac{4}{1 - c_0^2}\,\tilde{\phi}_{0,n}^{+}\frac{\mathrm{d}\tilde{\phi}_{0,n}^{+}}{\mathrm{d}\xi}\bigg|_{\xi=\xi_0}}$$



高次オーダー固有値（壁面上渦層）

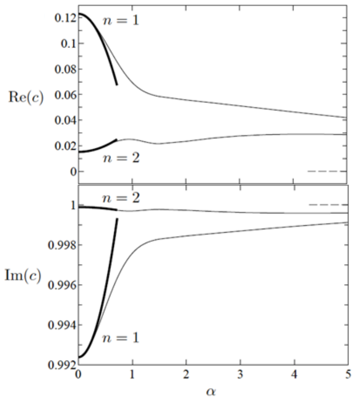
$n = 1$

sinuous　　varicose



$\mathrm{Im}(c_2) = 0.02(0.02)0.12$　　$\mathrm{Im}(c_2) = -0.04(0.02)0.16$

漸近解と数値解の比較



短波長
K-H不安定

5．乱流遷移及び乱流への力学系的アプローチ

乱流現象
非線形性，エネルギー散逸，巨大自由度

乱流構造の動力学
非再現性

乱流統計量
未完結性

乱流現象
非線形性，エネルギー散逸，巨大自由度

カオス的動力学
非再現性

関連づけ

普遍統計法則
未完結性

再現性　　厳密性

不変集合：固定点，周期軌道

Nagata (1990); Itano & Toh (2001); K. & Kida (2001);
Waleffe (2003); Faisst & Eckhardt (2003);
Wedin & Kerswell (2004); van Veen, Kida & K. (2006)

乱流現象
非線形性，エネルギー散逸，巨大自由度

カオス的動力学
非再現性

関連づけ

普遍統計法則
未完結性

再現性　　厳密性

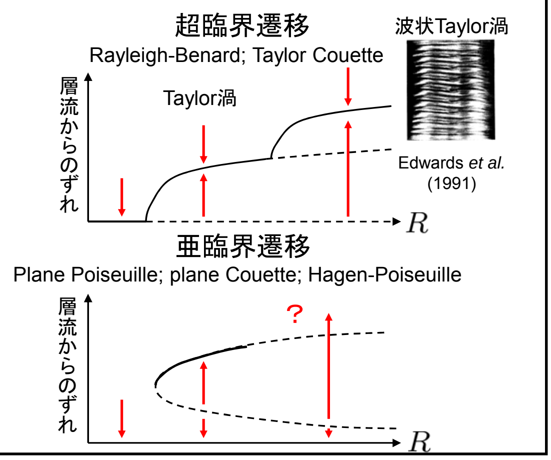
不変集合：固定点，周期軌道

Nagata (1990); Itano & Toh (2001); K. & Kida (2001);
Waleffe (2003); Faisst & Eckhardt (2003);
Wedin & Kerswell (2004); van Veen, Kida & K. (2006)

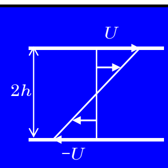
壁面剪断流の乱流遷移
Navier－Stokes 方程式系の固定点

- 平面Couette流
- 平面Poiseuille流
- Hagen-Poiseuille 流

IUTAM Symposium on
Laminar-Turbulent Transition
and Finite-Amplitude Solutions
(2004)
Bristol

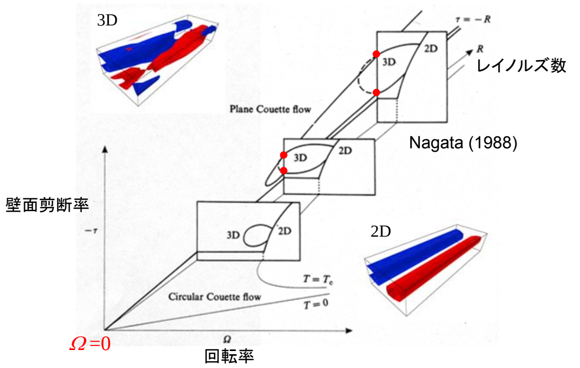


平面Couette流



- 基本流
  2次元の1次関数速度分布
- 線形安定性解析
  全てのReynolds数で安定
  基本流からの非線形解の分岐が不可
- 遷移Reynolds数
  $Re \fallingdotseq 300$

3次元定常解 （Nagata 1990）



サドル・ノード分岐（クエット流）

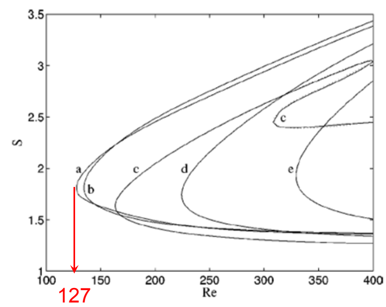
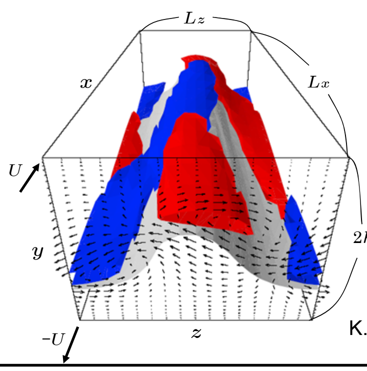


FIG. 19. Bifurcation diagrams for rigid–rigid Couette. Wall shear rate, S, normalized by its laminar value. (a) $(\alpha,\gamma)=(0.5772,1.1506)$, $[L_T,M_T,N_T]=[15,27,15]$ and $[13,25,13]$ overlap. (b) $(\alpha,\gamma)=(0.75,1.5)$, $[13,25,13]$ and $[11,13,11]$ overlap. (c) $(\alpha,\gamma)=(0.5,1.5)$, $[15,27,15]$ and $[13,25,13]$ (dot). (d) $(\alpha,\gamma)=(0.75,2.25)$, $[13,27,13]$. (e) $(\alpha,\gamma)=(1,3)$, $[13,27,13]$.

Waleffe （2003）

下分枝の解の空間構造

$Re=600,\ L_x/h=2\pi,\ L_z/h=0.8\pi$

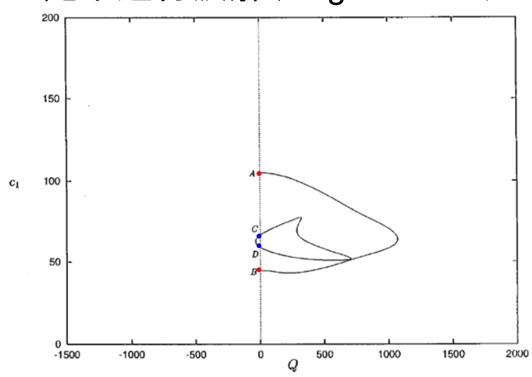


赤，青色等値面
$\omega_x=\pm 0.13U/h$

灰色等値面
$u=-0.2U$

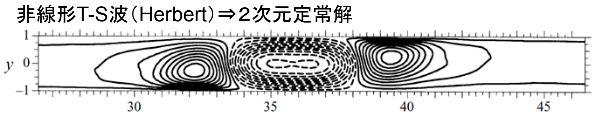
K. & Shiba (2004)

定常進行波解（Nagata 1997）

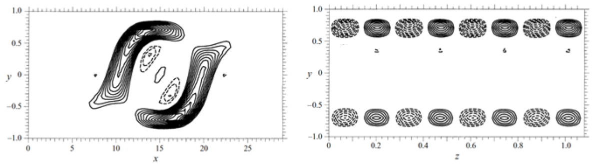


局在定常解？
（Cherhabili & Ehrenstein 1997）

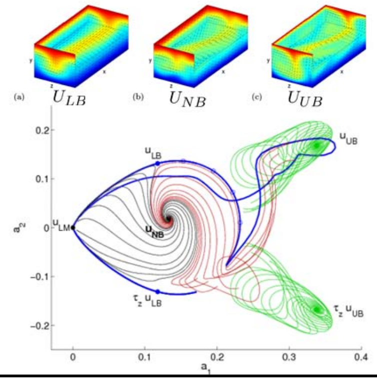
非線形T-S波（Herbert）⇒2次元定常解



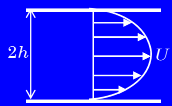
2次元定常解⇒3次元定常解



相空間の大域的構造（Couette流）
Gibson, Halcrow & Cvitanovic' (2008)

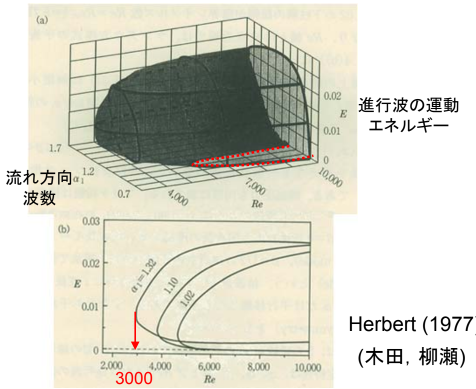


平面Poiseuille流



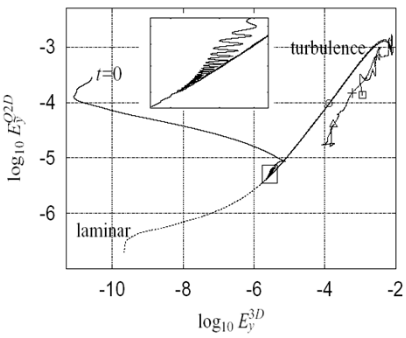
- 基本流
  2次元の2次関数速度分布
- 線形安定性解析
  $Re = Uh/\nu = 5772,\ \alpha h = 1.02$
  基本流からの2次元定常進行波の分岐
- 遷移Reynolds数
  $Re \fallingdotseq 1000$

2次元定常進行波

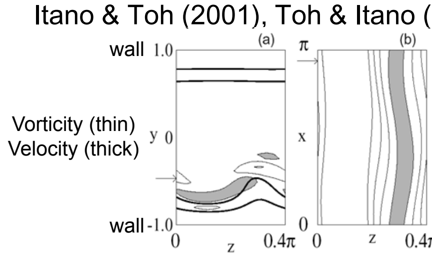


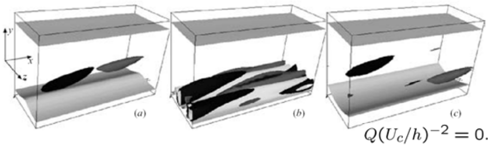
Herbert (1977)
(木田，柳瀬)

非対称3次元定常進行波
Toh & Itano (1999)



Itano & Toh (2001), Toh & Itano (2003)





$Q(U_c/h)^{-2} = 0.005$
$Q = (\omega_{i,j}\omega_{i,j} - e_{i,j}e_{i,j})/2$

対称3次元定常進行波
Waleffe (2003)

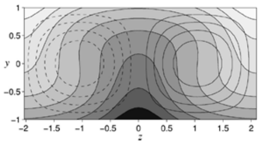


FIG. 4. Streaky flow $\mathbf{U}(y,z)$ for $F_r=5$, $\gamma=1.5$. Shaded contours of $U(y,z)$ at multiples of 0.1(max $U-$min U)=0.1703 with contours of ω_x at multiples of 0.2 max $\omega_x=0.0208$. Positive ω_x contours solid, negative dashed.

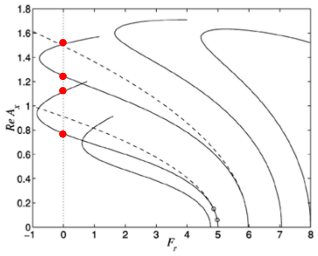


FIG. 6. Bifurcations from streaky flow in free-free plane Couette flow for $\gamma=1.5$, Re=150, and $\alpha=0.4$, 0.49, 0.64, 0.72, 0.76 originate at $A_x=0$ near $F_r=4.76$, 5, 6, 7, and 8, respectively. Dashed lines: $F_r=5-6.03(\mathrm{Re}\,A_x)^2$ and $F_r=5.98-2.68(\mathrm{Re}\,A_x)^2$. Open circles on $\alpha=0.49$ curve correspond to Fig. 7.

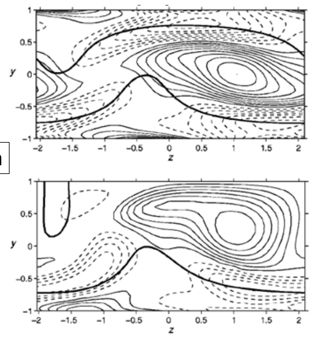
Waleffe (2003)

Couette system
(full height)

continuous connection

Poiseuille system
(half height)

Streamwise vorticity



対称3次元定常進行波
Waleffe (2003)

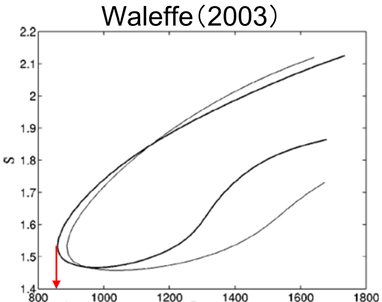
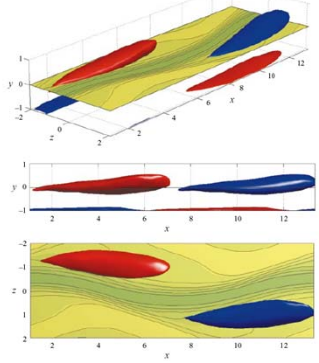


FIG. 17. Bifurcation diagrams for rigid–free Poiseuille. Wall shear rate, S, normalized by its laminar value vs bulk Reynolds number $\mathrm{Re}_m=\mathcal{U}H/\nu$ with $\nu=\langle U-U_{\mathrm{wall}}\rangle$ and where $\mathcal{U}=\langle U-U_{\mathrm{wall}}\rangle$ is the bulk mean velocity and $H=4\ell$ is the full channel height. $\mathrm{Re}_m=4\,\mathcal{U}\,\mathrm{Re}$ and $S=4/(3\mathcal{U})$ in our units. Near optimum $(\alpha,\gamma)=(0.507,1.31)$ (thick curve) and $(\alpha,\gamma)=(0.5,1.5)$, $[L_T,M_T,N_T]=[13,25,13]$.

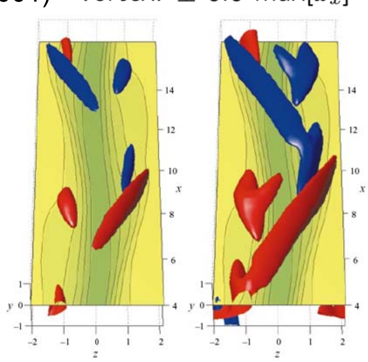
Waleffe (2001)　vortex: ± 0.6 max$[\omega_x]$

Upper branch

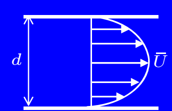


Waleffe (2001)　vortex: ± 0.6 max$[\omega_x]$

Lower branch



Hagen－Poiseuille流



- 基本流
  - 軸対称の2次関数速度分布
- 線形安定性解析
  - 全てのReynolds数で安定
  - 基本流からの非線形解の分岐が不可
- 遷移Reynolds数
  - $Re = \bar{U}d/\nu \fallingdotseq 2000$

3次元定常進行波

Faisst & Eckhardt (2003); Wedin & Kerswell (2004)

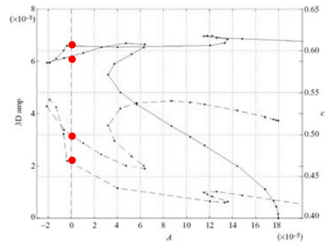


FIGURE 5. The continuation curve for $m_0 = 3, \alpha = 2.44$ at $Re = 1800$ ($A_{crit} = 1.7962 \times 10^{-4}$ as opposed to $A_{crit} = 1.75 \times 10^{-4}$ predicted from the SSP analysis: truncation (8, 24, 5)). Again there are two crossings of the pipe flow limit ($A = 0$) representing an upper and lower branch of travelling wave solutions.

Wedin & Kerswell (2004)

3次元定常進行波

Faisst & Eckhardt (2003); Wedin & Kerswell (2004)

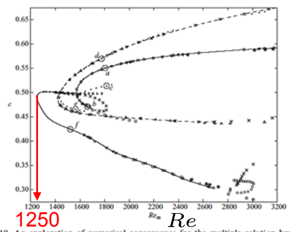
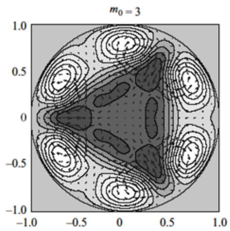


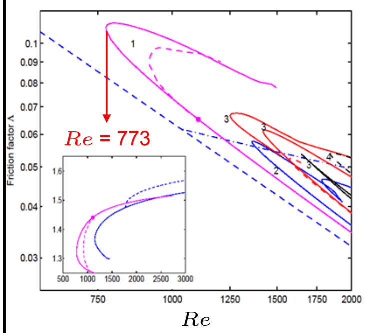
FIGURE 10. An exploration of numerical convergence for the multiple solution branches for $\mathscr{R}_3$ revealed in figure 7. The solutions a–f at $Re = 2200$ are marked as in figure 7 but now over Re_m. A solid or dashed line indicates where a solution branch has been confirmed by different truncation levels (the dashed line corresponds to the dashed line in figure 7). This shows that the (8, 24, 5) solutions a, b, d, e and f are real (although e is poorly resolved) and that c is a numerical artefact. Notice that (9, 26, 6) indicates that there are two separate branches. The truncation levels (M, N, L) used are ● (8, 24, 5), • (8, 24, 6), □ (8, 24, 7), ◇ (8, 26, 5), × (9, 26, 6), △ (8, 28, 6), ○ (9, 28, 5).



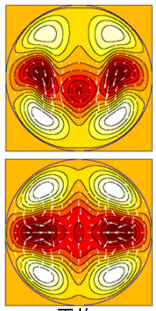
Wedin & Kerswell (2004)

非対称定常進行波（Hagen-Poiseuille流）

Pringle & Kerswell (2007)



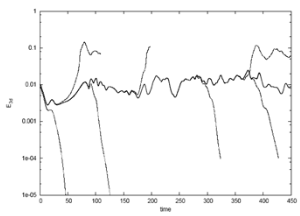
鏡像対称解



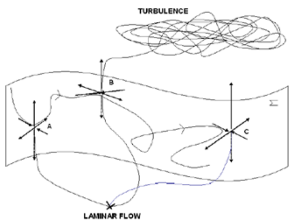
平均

非対称定常進行波（Hagen-Poiseuille流）

Duguet, Willis & Kerswell (2008)

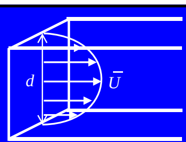


Edge of chaos
(Skufca, Yorke, Eckhardt 2006)



ヘテロクリニック接続？

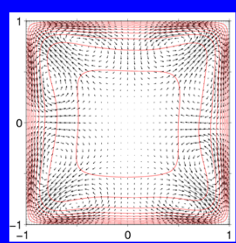
正方形ダクト流

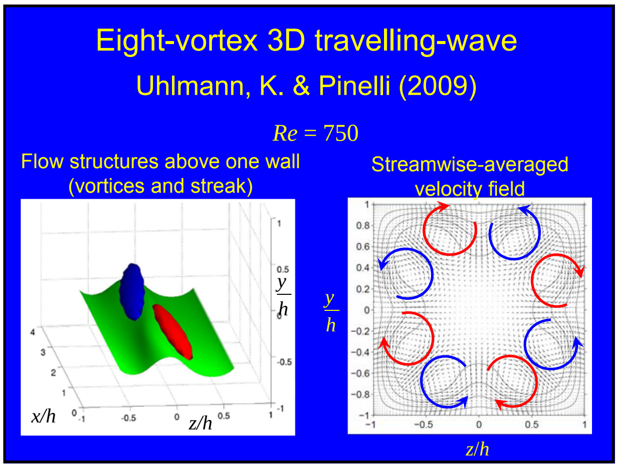


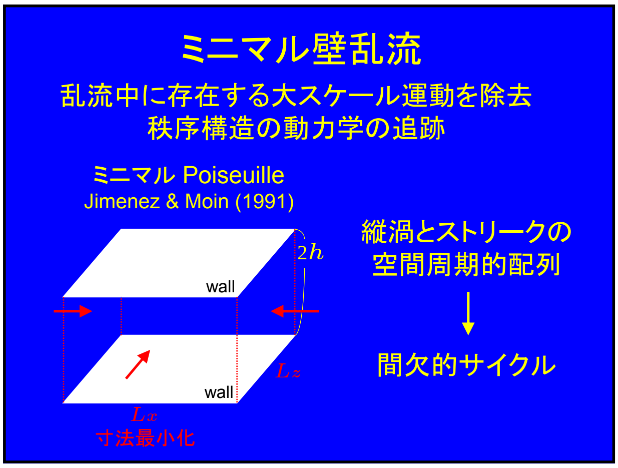
- 基本流
  三角関数，双曲線関数
- 線形安定性解析
  全てのReynolds数で安定
  基本流からの非線形解の分岐が不可
  臨界アスペクト比 3.2 (Tatsumi & Yoshimura 1990)
- 遷移Reynolds数
  $Re = \bar{U}d/\nu \fallingdotseq 2000$
  (Uhlmann, Pinelli, K. & Sekimoto 2007)

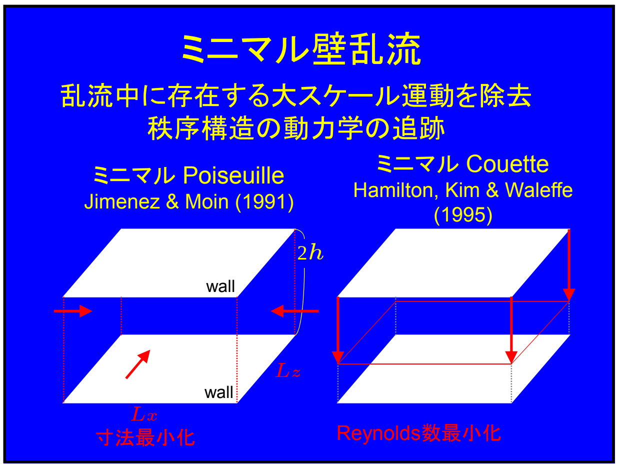
Turbulence-driven secondary flow
Prandtl's second kind

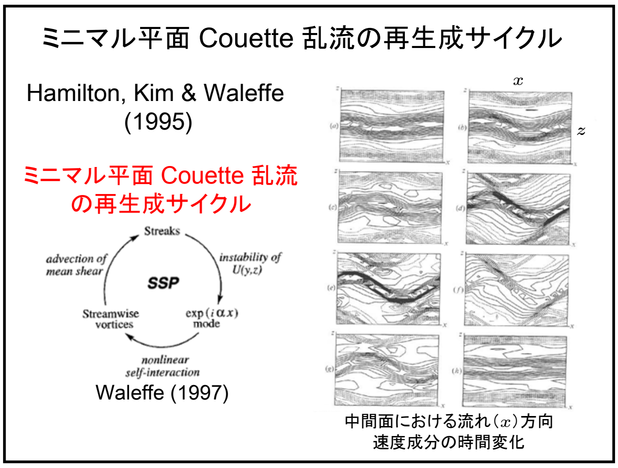
Generation mechanism
- Statistical budget
  Kajishima, Miyake & Nishimoto (1991)
- Transient growth
  Biau, Soueid & Bottaro (2008)
- Coherent structures
  Uhlmann, Pinelli, Sekimoto & K. (2010)

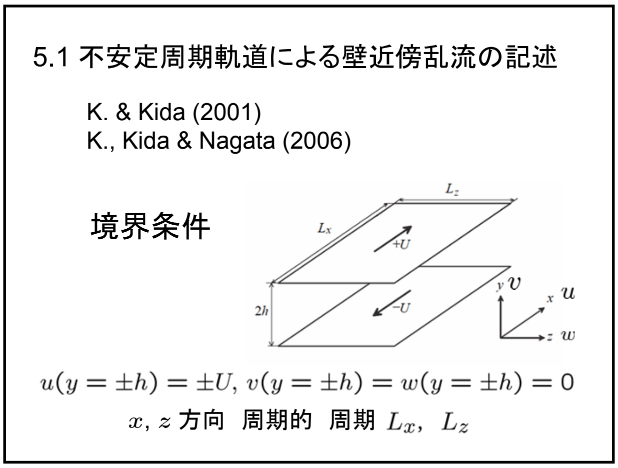


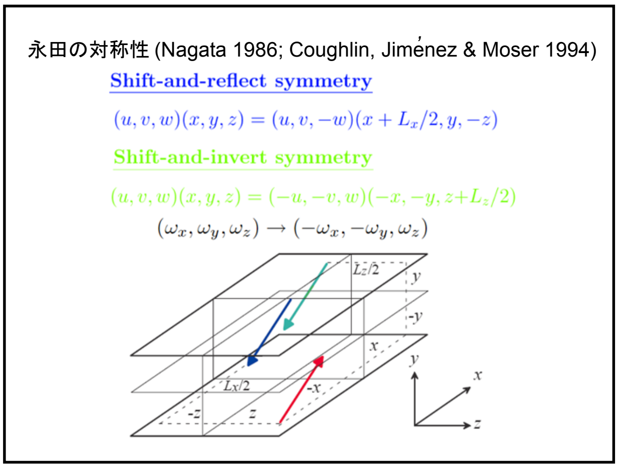

Eight-vortex 3D travelling-wave
Uhlmann, K. & Pinelli (2009)
Re = 750
Flow structures above one wall
(vortices and streak)
Streamwise-averaged
velocity field
x/h
z/h
y/h


ミニマル壁乱流
乱流中に存在する大スケール運動を除去
秩序構造の動力学の追跡
ミニマル Poiseuille
Jimenez & Moin (1991)
縦渦とストリークの
空間周期的配列
間欠的サイクル
wall
wall
2h
Lz
Lx
寸法最小化


ミニマル壁乱流
乱流中に存在する大スケール運動を除去
秩序構造の動力学の追跡
ミニマル Poiseuille
Jimenez & Moin (1991)
ミニマル Couette
Hamilton, Kim & Waleffe
(1995)
wall
wall
2h
Lz
Lx
寸法最小化
Reynolds数最小化


ミニマル平面 Couette 乱流の再生成サイクル
Hamilton, Kim & Waleffe
(1995)
ミニマル平面 Couette 乱流
の再生成サイクル
Streaks
advection of mean shear
instability of U(y,z)
SSP
Streamwise vortices
exp(iαx) mode
nonlinear self-interaction
Waleffe (1997)
x
z
中間面における流れ(x)方向
速度成分の時間変化


5.1 不安定周期軌道による壁近傍乱流の記述
K. & Kida (2001)
K., Kida & Nagata (2006)
境界条件
Lz
Lx
+U
-U
2h
y v
x u
z w
u(y = ±h) = ±U, v(y = ±h) = w(y = ±h) = 0
x, z 方向　周期的　周期 Lx, Lz


永田の対称性 (Nagata 1986; Coughlin, Jiménez & Moser 1994)
Shift-and-reflect symmetry
(u, v, w)(x, y, z) = (u, v, −w)(x + Lx/2, y, −z)
Shift-and-invert symmetry
(u, v, w)(x, y, z) = (−u, −v, w)(−x, −y, z+Lz/2)
(ωx, ωy, ωz) → (−ωx, −ωy, ωz)
Lz/2
Lx/2
y
-y
x
-x
z
-z

非圧縮 Navier-Stokes 方程式

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u}$$

$$\nabla\cdot\boldsymbol{u} = 0$$

$$Re = Uh/\nu$$

無次元化された境界条件

$$u(y=\pm 1) = \pm 1,\ v(y=\pm 1) = w(y=\pm 1) = 0$$

圧力の消去

平行平板間流の直接数値シミュレーション

Kim, Moin & Moser (1987)

Navier-Stokes 方程式の rot の壁面垂直方向への射影

$$\left(\frac{\partial}{\partial t} - \frac{1}{Re}\nabla^2\right)\omega_y = -\boldsymbol{e}_y\cdot[\nabla\times(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]$$

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{u} = (\omega_x, \omega_y, \omega_z)^{\mathrm{t}}$$

Navier-Stokes 方程式の2回 rot の壁面垂直方向への射影

$$\left(\frac{\partial}{\partial t} - \frac{1}{Re}\nabla^2\right)\nabla^2 v = \boldsymbol{e}_y\cdot[\nabla\times\nabla\times(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]$$

Fourier 展開

$$f(x,y,z;t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\tilde{f}_{m,n}(y;t)\,\mathrm{exp}\,[\mathrm{i}\,(\alpha m x + \beta n z)]$$

$$\tilde{f}_{m,n} = \frac{1}{L_x L_z}\int_0^{L_x}\int_0^{L_z} f\,\mathrm{exp}\,[-\mathrm{i}\,(\alpha m x + \beta n z)]\,\mathrm{d}x\mathrm{d}z$$

$$\alpha = 2\pi/L_x,\ \ \beta = 2\pi/L_z$$

Fourier 係数に対する方程式

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{1}{Re}\mathcal{L}_{m,n}\right)\tilde{\omega_y}_{m,n} &= \tilde{g}_{m,n} \\ \left(\frac{\partial}{\partial t} - \frac{1}{Re}\mathcal{L}_{m,n}\right)\mathcal{L}_{m,n}\tilde{v}_{m,n} &= \tilde{h}_{m,n}\end{aligned}\right\}\text{擬スペクトル法}$$

$$\mathcal{L}_{m,n} = \frac{\partial^2}{\partial y^2} - (\alpha m)^2 - (\beta n)^2$$

u, w の計算

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = -\frac{\partial v}{\partial y} \longrightarrow \mathrm{i}\alpha m\tilde{u}_{m,n} + \mathrm{i}\beta n\tilde{w}_{m,n} = -\frac{\partial \tilde{v}_{m,n}}{\partial y}$$

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \omega_y \longrightarrow \mathrm{i}\beta n\tilde{u}_{m,n} - \mathrm{i}\alpha m\tilde{w}_{m,n} = \tilde{\omega_y}_{m,n}$$

ゼロモードの方程式

$$\frac{\partial \tilde{u}_{0,0}}{\partial t} = -\frac{\partial}{\partial y}\widetilde{uv}_{0,0} + \frac{1}{Re}\frac{\partial^2 \tilde{u}_{0,0}}{\partial y^2}$$

Fourier 係数に対する境界条件

$$\tilde{\omega_y}_{m,n}(y=\pm 1) = \tilde{v}_{m,n}(y=\pm 1) = \frac{\partial \tilde{v}_{m,n}}{\partial y}(y=\pm 1) = 0$$

$$\tilde{u}_{0,0}(y=\pm 1) = \pm 1$$

時間積分:Adams-Bashforth 法＋Crank-Nicolson 法

$$\left(1 - \frac{\Delta t}{2Re}\mathcal{L}_{m,n}\right)\tilde{\omega_y}_{m,n}^{k+1} = \frac{\Delta t}{2}(3\tilde{g}_{m,n}^{k} - \tilde{g}_{m,n}^{k-1}) + \left(1 + \frac{\Delta t}{2Re}\mathcal{L}_{m,n}\right)\tilde{\omega_y}_{m,n}^{k}$$

$$\left(1 - \frac{\Delta t}{2Re}\mathcal{L}_{m,n}\right)\mathcal{L}_{m,n}\tilde{v}_{m,n}^{k+1} = \frac{\Delta t}{2}(3\tilde{h}_{m,n}^{k} - \tilde{h}_{m,n}^{k-1}) + \left(1 + \frac{\Delta t}{2Re}\mathcal{L}_{m,n}\right)\mathcal{L}_{m,n}\tilde{v}_{m,n}^{k}$$

$$\left(1 - \frac{\Delta t}{2Re}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right)\tilde{u}_{0,0}^{k+1} = -\frac{\Delta t}{2}\frac{\mathrm{d}}{\mathrm{d}y}(3\widetilde{uv}_{0,0}^{k} - \widetilde{uv}_{0,0}^{k-1}) + \left(1 + \frac{\Delta t}{2Re}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right)\tilde{u}_{0,0}^{k}$$

変形 Chebyshev 展開（McFadden, Murray & Boisvert 1990）

$$\tilde{\omega}_{m,n}(y,t) = \sum_{l=2}^{\infty}\hat{\tilde{\omega}}_{m,n,l}(t)\chi_l(y)$$

$$\tilde{v}_{m,n}(y,t) = \sum_{l=4}^{\infty}\hat{\tilde{v}}_{m,n,l}(t)\psi_l(y)$$

$$\tilde{u}_{0,0}(y,t) = \sum_{l=2}^{\infty}\hat{\tilde{u}}_{0,0,l}(t)\chi_l(y) + T_1(y)$$

$$\chi_{2k}(y) = T_{2k}(y) - T_0(y)$$
$$\chi_{2k+1}(y) = T_{2k+1}(y) - T_1(y)$$
$$(k = 1, 2, 3, \cdots)$$

$$\psi_{2k}(y) = T_{2k}(y) + (k^2-1)T_0(y) - k^2 T_2(y)$$
$$\psi_{2k+1}(y) = T_{2k+1}(y) + \tfrac{1}{2}(k^2+k-2)T_1(y) - \tfrac{1}{2}(k^2+k)T_3(y)$$
$$(k = 2, 3, \cdots)$$

相空間における流れ

Fourier-Chebyshev 係数

$$\underbrace{\{\hat{\hat{u}}_{0,0,l}\}\ \{\mathrm{Re}(\hat{\hat{v}}_{m,n,l})\}\{\mathrm{Im}(\hat{\hat{v}}_{m,n,l})\}\ \{\mathrm{Re}(\hat{\hat{\omega}}_{y_{m,n,l}})\}\{\mathrm{Im}(\hat{\hat{\omega}}_{y_{m,n,l}})\}}_{N\text{ 次元ベクトル}}$$

N 次元相空間の状態点

$$\boldsymbol{x} = (x_1, x_2, \cdots, x_N)^{\mathrm{t}}$$

力学系

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(\boldsymbol{x}; Re) \longleftarrow$$ Navier-Stokes 方程式で与えられるベクトル場

流れ

$$\boldsymbol{\phi} = (\phi_1, \phi_2, \cdots, \phi_N)^{\mathrm{t}}$$

$$x_i = \phi_i(\boldsymbol{x}_0, t) \quad (x_i = \phi_i(\boldsymbol{x}_0, 0) = x_{0i})$$

不安定周期軌道の数値計算

周期軌道（周期 T）

$$\phi_i(\boldsymbol{x}, T) = x_i$$

Poincaré 断面

$$x_N = \text{const.}$$

Poincaré 断面における解の補正

推定解 $(\boldsymbol{r}, T)$ $\quad \boldsymbol{r} = (r_1, r_2, \cdots, r_{N-1})^{\mathrm{t}}$

補正 $(\boldsymbol{s}, \delta T)$ $\quad \boldsymbol{s} = (s_1, s_2, \cdots, s_{N-1})^{\mathrm{t}}$

$$\phi_i(\boldsymbol{r}+\boldsymbol{s}, x_N, T+\delta T) = r_i + s_i \quad (i \le N-1)$$

$$\phi_N(\boldsymbol{r}+\boldsymbol{s}, x_N, T+\delta T) = x_N$$

線形化方程式

$$\sum_{j=1}^{N-1}\left[\frac{\partial \phi_i}{\partial x_j}(\boldsymbol{r}, x_N, T) - \delta_{ij}\right] s_j + \frac{\partial \phi_i}{\partial t}(\boldsymbol{r}, x_N, T)\delta T = -\left[\phi_i(\boldsymbol{r}, x_N, T) - r_i\right] \quad (i \le N-1)$$

$$\sum_{j=1}^{N-1}\left[\frac{\partial \phi_N}{\partial x_j}(\boldsymbol{r}, x_N, T) - \delta_{ij}\right] s_j + \frac{\partial \phi_N}{\partial t}(\boldsymbol{r}, x_N, T)\delta T = -\left[\phi_N(\boldsymbol{r}, x_N, T) - x_N\right]$$

解の更新

$$(\boldsymbol{r}, T) \to (\boldsymbol{r}+\boldsymbol{s}, T+\delta T)$$

微分係数の計算

基準軌道 $\phi_i(\boldsymbol{x}, t)\ (t > 0)$

線形化された流れ $\delta\phi_i(\delta\boldsymbol{x}, t)\ \ (\delta x_i = \delta\phi_i(\delta\boldsymbol{x}, 0))$

$$\phi_i(\boldsymbol{x}+\boldsymbol{\delta} x, t) = \phi_i(\boldsymbol{x}, t) + \delta\phi_i(\delta\boldsymbol{x}, t) = \phi_i(\boldsymbol{x}, t) + \frac{\partial \phi_i}{\partial x_k}(\boldsymbol{x}, t)\delta x_k$$

$$\frac{\partial \phi_i}{\partial x_j}(\boldsymbol{r}, x_N, T) = \delta\phi_i(\delta x_k = \delta_{kj}, T)\ (j \le N-1)$$

$$\frac{\partial \phi_i}{\partial t}(\boldsymbol{r}, x_N, T) = f_i(\boldsymbol{\phi}(\boldsymbol{r}, x_N, T))$$

（$\{\hat{\hat{v}}_{m,n,l}\}$ に対しては有限差分）

N 元連立1次方程式

$$A\boldsymbol{y} = \boldsymbol{b}$$

$$A = \begin{pmatrix} \dfrac{\partial\phi_1}{\partial x_1} - 1 & \dfrac{\partial\phi_1}{\partial x_2} & \cdots & \dfrac{\partial\phi_1}{\partial x_{N-1}} & \dfrac{\partial\phi_1}{\partial t} \\ \dfrac{\partial\phi_2}{\partial x_1} & \dfrac{\partial\phi_2}{\partial x_2} - 1 & \cdots & \dfrac{\partial\phi_2}{\partial x_{N-1}} & \dfrac{\partial\phi_2}{\partial t} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial\phi_N}{\partial x_1} & \dfrac{\partial\phi_N}{\partial x_2} & \cdots & \dfrac{\partial\phi_N}{\partial x_{N-1}} & \dfrac{\partial\phi_N}{\partial t} \end{pmatrix}$$

$$\boldsymbol{y} = (s_1, s_2, \cdots, s_{N-1}, \delta T)^{\mathrm{t}}$$

$$\boldsymbol{b} = -(\phi_1 - r_1, \phi_2 - r_2, \cdots, \phi_N - x_N)^{\mathrm{t}}$$

連立1次方程式の解法

- 直接法 $N = O(10^4)$
  Gauss の消去法
  - K. (2005)
  - K., Kida & Nagata (2006)
  - van Veen, Kida & K. (2006)
- 反復法 $N = O(10^5)$
  GMRES（generalized minimal residuals）
  - Viswanath (2007)
  - Duguet (2007)

GMRES
Saad & Schultz (1986)

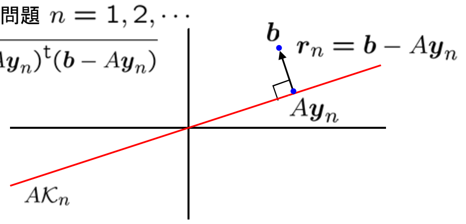
Krylov 部分空間

$\mathcal{K}_n = \mathrm{span}(\boldsymbol{b}, A\boldsymbol{b}, \cdots, A^{n-1}\boldsymbol{b})$ $\quad \boldsymbol{y}_n \in \mathcal{K}_n$

$A\mathcal{K}_n = \mathrm{span}(A\boldsymbol{b}, A^2\boldsymbol{b}, \cdots, A^n\boldsymbol{b})$ $\quad A\boldsymbol{y}_n \in A\mathcal{K}_n$

最小2乗問題 $n = 1, 2, \cdots$

$\sqrt{(\boldsymbol{b} - A\boldsymbol{y}_n)^{\mathrm{t}}(\boldsymbol{b} - A\boldsymbol{y}_n)}$



初期推定

- 回帰的ふるまいの探索
  乱流軌道の計算と長時間軌道断片の記憶
  低 Reynolds 数から高 Reynolds 数へ
- カオス制御(不安定周期解の安定化)
  遅延 feedback 法　Pyragas (1992)

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(\boldsymbol{x}) + k[x_i(t-\tau) - x_i(t)]$$

奇数制約

大域的方法

残差

$$r(\boldsymbol{r}, T) = \sqrt{[\phi(\boldsymbol{r}, x_N, T) - \boldsymbol{x}]^{\mathrm{t}}[\phi(\boldsymbol{r}, x_N, T) - \boldsymbol{x}]}$$

が減少するように推定解を取得

- backtracking line-search
- locally constrained optimal hook step
- double dogleg step

(Dennis & Schnabel 1996)

DNS (藤 1999) Kim, Moin & Moser (1987)

- Fourier−Chebyshev スペクトル法
- 格子点数 4352 (= 16 × 17 × 16 in x, y, z) 33
  $N = 1402$

パラメター Hamilton, Kim & Waleffe (1995)

- $Re = Uh/\nu = 400$, $Re_\tau = u_\tau h/\nu = 34$
- $L_x = 1.755\pi h$, $L_x^+ = 188\nu/u_\tau$
- $L_z = 1.2\pi h$, $L_z^+ = 129\nu/u_\tau$
- $\Delta x^+ = 12$, $\Delta y^+ = 0.7 - 6.6$, $\Delta z^+ = 8$
  0.1—3.3

$$\frac{r(\boldsymbol{r}, T)}{\sqrt{\boldsymbol{x}^{\mathrm{t}}\boldsymbol{x}}} < 10^{-7}$$

エネルギー注入率 I

エネルギー散逸率 D

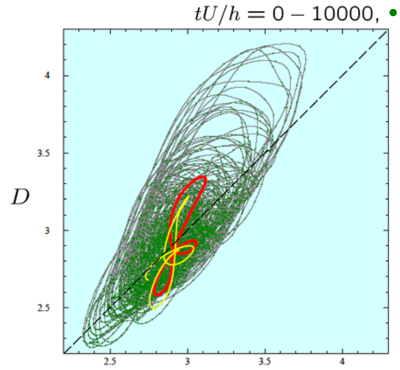
$$I = \frac{h}{2L_xL_zU}\int_0^{L_x}\int_0^{L_z}\left(\left.\frac{\partial u}{\partial y}\right|_{y=-h} + \left.\frac{\partial u}{\partial y}\right|_{y=+h}\right)\mathrm{d}x\mathrm{d}z$$

$$D = \frac{h}{2L_xL_zU^2}\int_0^{L_x}\int_{-h}^{+h}\int_0^{L_z}|\omega|^2\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

u, 流れ(x)方向速度

ω, 渦度

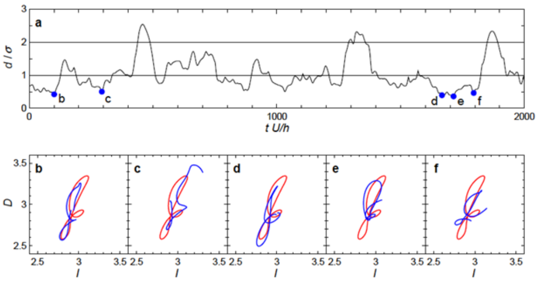
$tU/h = 0 - 10000$, • $\delta tU/h = 2$



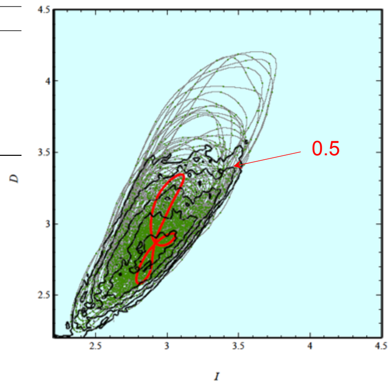
K. & Kida (2001) $\quad I$

周期 $T = 64.7h/U = 188\nu/u_\tau^2$

$$d(t) = \min_{\tau_0} \sqrt{\frac{1}{T}\int_0^T \left[\left(I_{\mathrm{T}}(t+\tau) - I_{\mathrm{P}}(\tau_0+\tau)\right)^2 + \left(D_{\mathrm{T}}(t+\tau) - D_{\mathrm{P}}(\tau_0+\tau)\right)^2\right] \mathrm{d}\tau}$$



prob.dens.	prob.
0.25	0.93
0.5	0.87
1.0	0.75
2.0	0.52
4.0	0.18

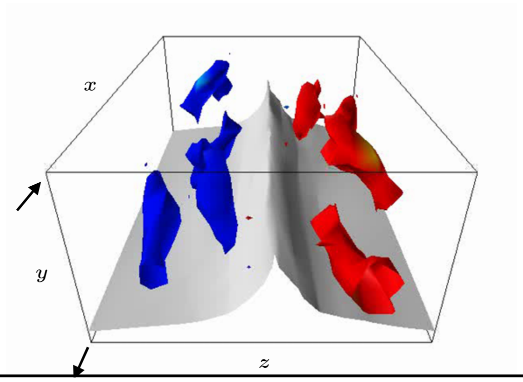


周期運動

赤，青等値面 $\nabla^2 p = 0.15\rho U^2/h^2$

（赤 時計回り $\omega_x > 0$，青 反時計回り $\omega_x < 0$）

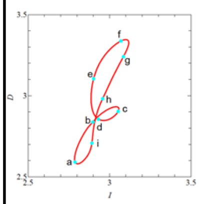
灰色等値面 $u = -0.3U$

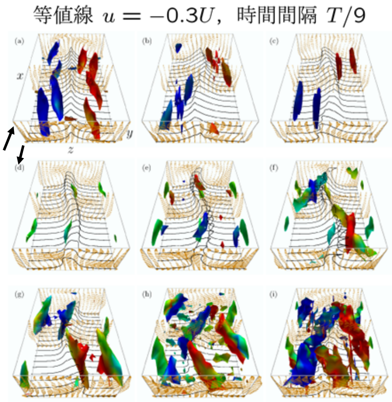


周期運動

赤，青等値面 $\nabla^2 p = 0.15\rho U^2/h^2$

等値線 $u = -0.3U$，時間間隔 $T/9$



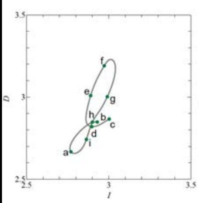


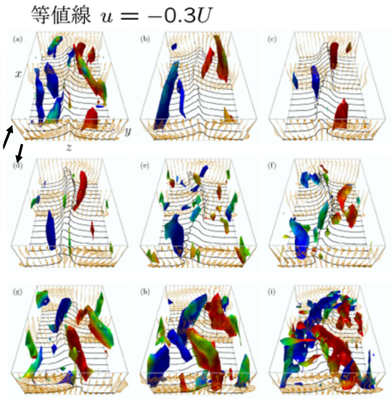
K. & Kida (2001)

乱流運動

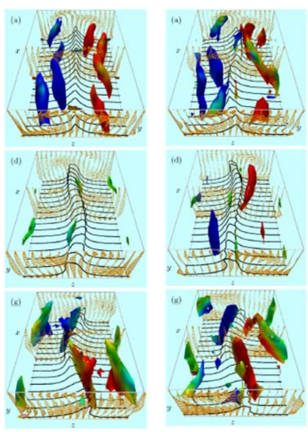
赤，青等値面 $\nabla^2 p = 0.15\rho U^2/h^2$

等値線 $u = -0.3U$

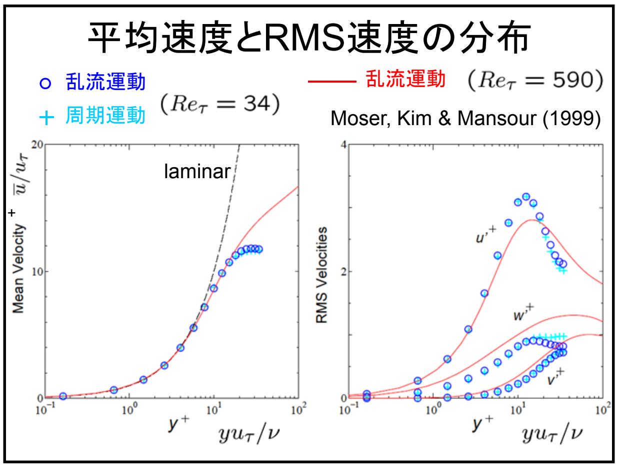


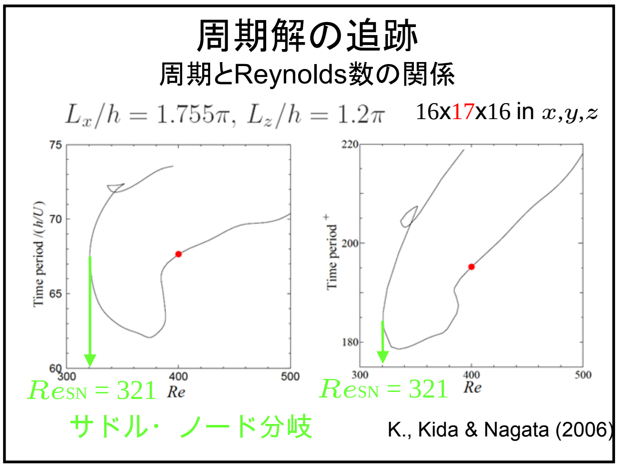


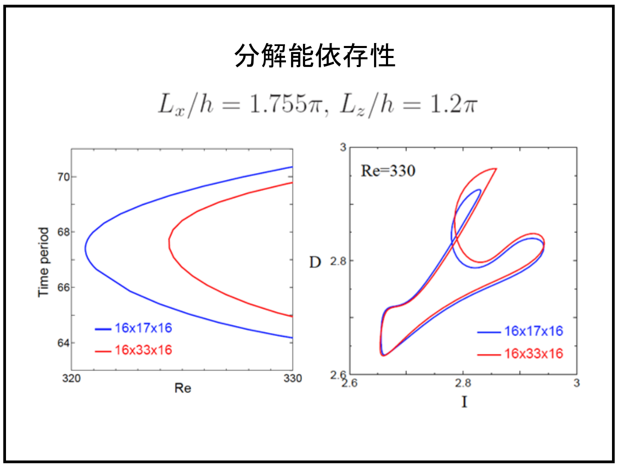
Periodic

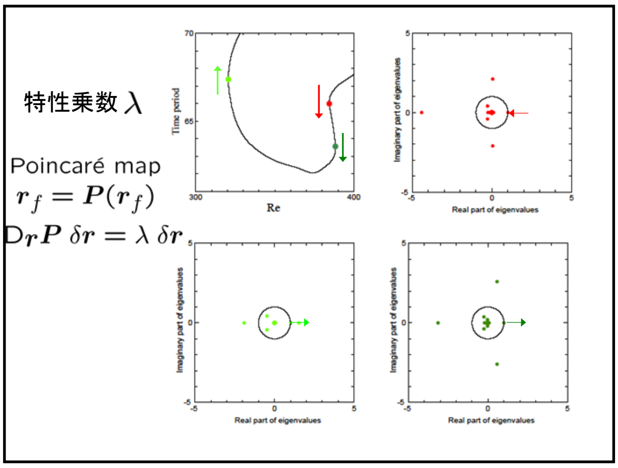


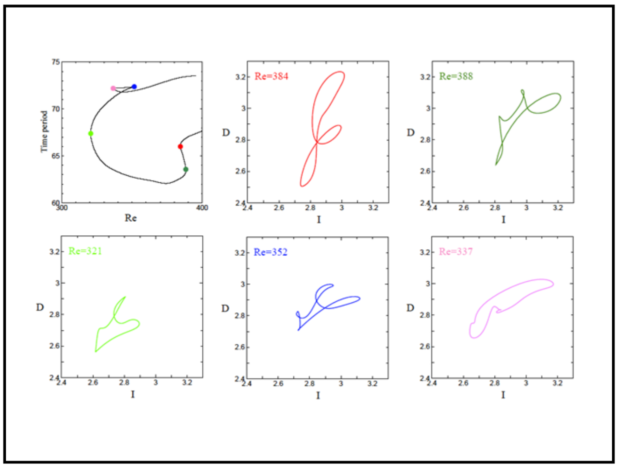
Turbulent

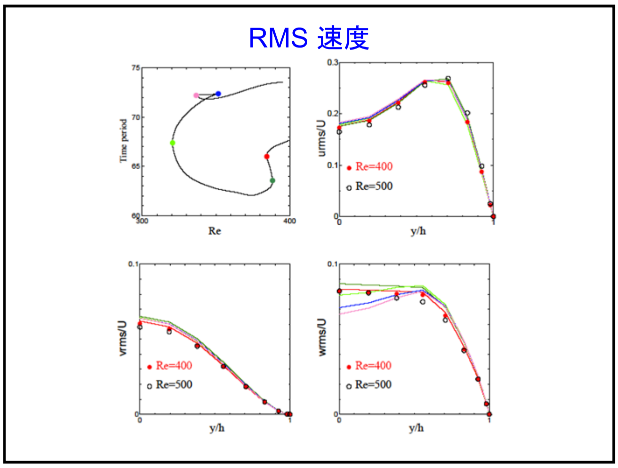

平均速度とRMS速度の分布
o 乱流運動
+ 周期運動
$(Re_\tau = 34)$
乱流運動 $(Re_\tau = 590)$
Moser, Kim & Mansour (1999)
laminar
Mean Velocity $\overline{u}/u_\tau$
RMS Velocities
u'^+
w'^+
v'^+
y^+ yu_τ/ν


周期解の追跡
周期とReynolds数の関係
$L_x/h = 1.755\pi,\ L_z/h = 1.2\pi$
16x17x16 in x, y, z
Time period $/(h/U)$
Time period$^+$
Re
$Re_{SN} = 321$
$Re_{SN} = 321$
サドル・ノード分岐
K., Kida & Nagata (2006)


分解能依存性
$L_x/h = 1.755\pi,\ L_z/h = 1.2\pi$
Time period
Re
Re=330
D
I
16x17x16
16x33x16

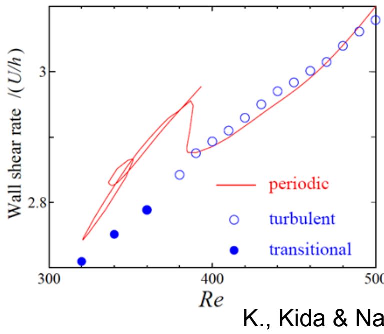

特性乗数 λ
Poincaré map
$r_f = P(r_f)$
$D_r P\, \delta r = \lambda\, \delta r$
Time period
Re
Imaginary part of eigenvalues
Real part of eigenvalues


Time period
Re
Re=384
Re=388
Re=321
Re=352
Re=337
D
I


RMS 速度
Time period
Re
urms/U
vrms/U
wrms/U
y/h
Re=400
Re=500

Wall shear rate vs. Reynolds number

$L_x/h = 1.755\pi,\ L_z/h = 1.2\pi$



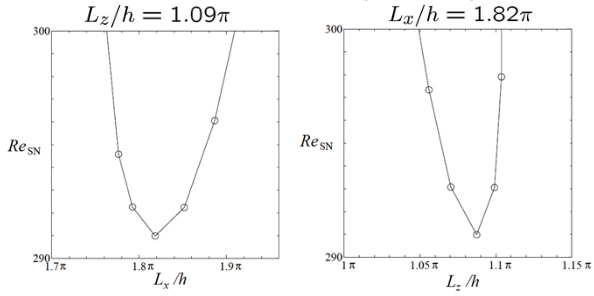
K., Kida & Nagata (2006)

The minimum Reynolds number

K., Kida & Nagata (2006)

$$Re_{\mathrm{SN}} = Re_{\mathrm{SN}}(L_x, L_z)$$

$L_z/h = 1.09\pi$ $L_x/h = 1.82\pi$

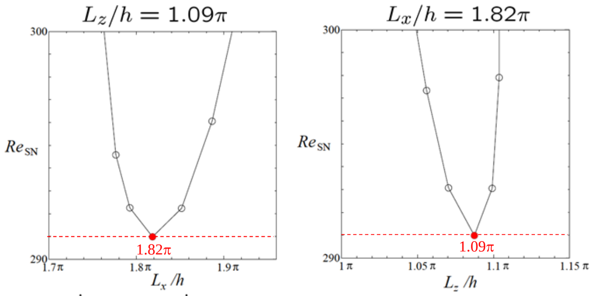


The minimum Reynolds number

K., Kida & Nagata (2006)

$Re_{\mathrm{SN}} = 291$ for $(L_x, L_z) = (1.82\pi, 1.09\pi)$

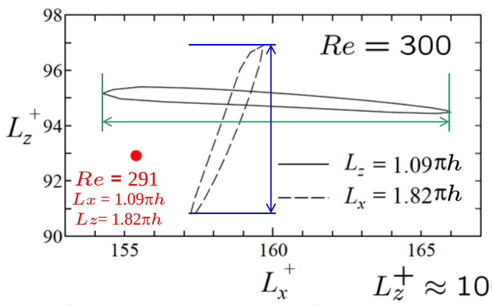
$L_z/h = 1.09\pi$ $L_x/h = 1.82\pi$



$(L_x^+, h^+, L_z^+) \simeq (155, 27, 93),\ T^+ \simeq 201$

周期箱寸法の選択

K., Kida & Nagata (2006)



$L_z^+ \approx 100$

$L_x^+ \simeq 154 - 166,\ L_z^+ \simeq 91 - 97$

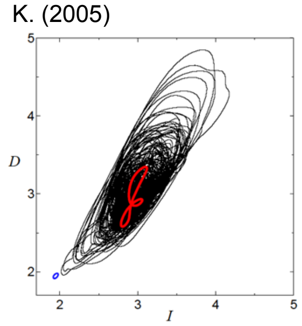
平面 Couette 流の乱流遷移

- Dauchot & Daviaud (1995)
  $A \sim (Re - Re_{NL})^{-a} \rightarrow Re_{NL} = 325 \pm 5$
  $a = 0.3 - 0.8$
- Bottin & Chaté (1998)
  $t \sim (Re_{NL} - Re)^{-1} \rightarrow Re_{NL} = 323 \pm 2$

$Re \approx 290$

5.2 不安定周期軌道による壁近傍乱流の制御

Jimenez, K., Simens, Nagata & Shiba (2005)
K. (2005)



$Re = 400$

Periodic
$\overline{I} = 1.95$
$T = 85.3h/U = 248\nu/u_\tau^2$

Turbulent
$\overline{I} = 2.91$
$T \approx 2.5 \times 10^4 h/U$
$\approx 7.2 \times 10^4 \nu/u_\tau^2$

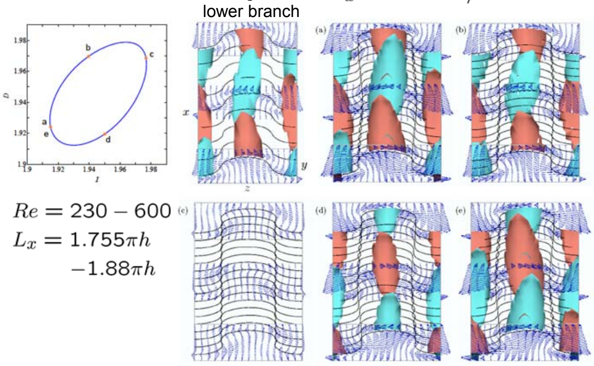
エネルギー注入率＝壁面剪断率

K. & Kida (2001); Jimenez, K., Simens, Nagata & Shiba (2005)

静穏周期解

Nagata lower branch

Streamwise vorticity $\omega_x = \pm 0.15U/h$

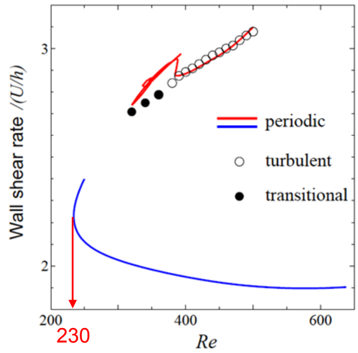


$Re = 230 - 600$
$L_x = 1.755\pi h$
$-1.88\pi h$

K. & Kida (2001); Jimenez, K., Simens, Nagata & Shiba (2005)

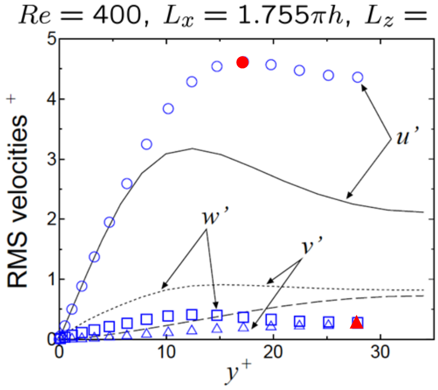
Wall shear rate vs. Reynolds number

$L_x/h = 1.755\pi,\ L_z/h = 1.2\pi$



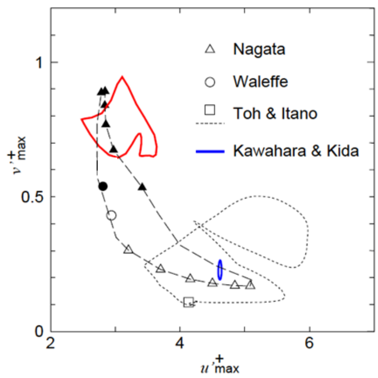
RMS 速度分布

$Re = 400,\ L_x = 1.755\pi h,\ L_z = 1.2\pi h$



線：乱流
記号：周期流

RMS速度u', v'の最大値

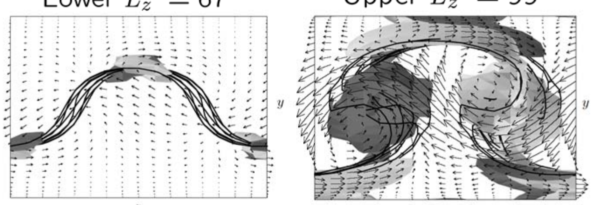


Jimenez, K., Simens, Nagata & Shiba (2005)

空間構造の違い（平衡解）

$Re = 400,\ L_x \times L_z = 2\pi h \times 0.9\pi h$

Lower $L_z^+ = 67$ Upper $L_z^+ = 99$



vector scale:$1.9u_\tau$; isosurfaces:$\omega_x^+ = \pm 0.155$

Jimenez, K., Simens, Nagata & Shiba (2005)

（亜臨界）壁乱流の制御

摩擦抵抗低減 → 層流化

- Bewley, Moin & Temam (2001)
  最適制御理論(非線形)
  ポアズイユ乱流 $Re_\tau = u_\tau h/\nu = 100$
- Hogberg, Bewley & Henningson (2003)
  最適制御理論（線形）
  ポアズイユ乱流 $Re_\tau = 100$

膨大な計算が必要

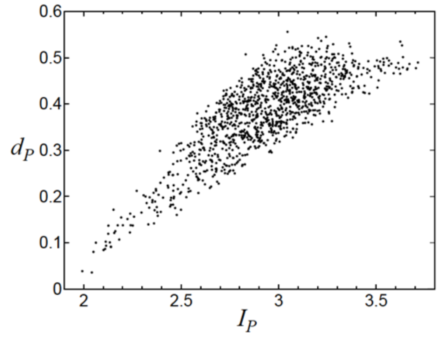
カオス制御

- カオスに内在する不安定周期軌道の安定化
  パラメターへの敏感な依存性
  → わずかな入力による制御
- カオス ⇒ （望ましい）周期運動
- 制御法
  - Ott,Grebogi & Yorke (1990) OGY法
  - Pyragas (1992) 外力制御

目標周期軌道を求めるのが困難

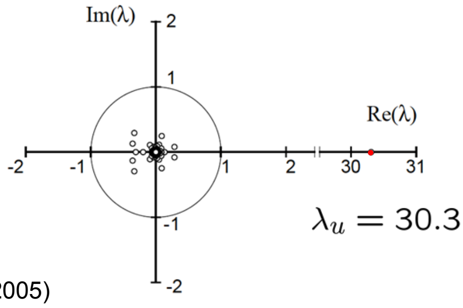
Poincare 断面での距離と壁面剪断率

$d_p = ||\boldsymbol{x} - \boldsymbol{x}_f|| / ||\boldsymbol{x}_f||$ (N dimension)



周期運動の安定性

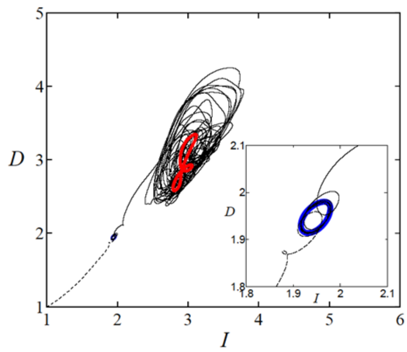
$\mathrm{D}_{\boldsymbol{r}}\boldsymbol{P}(\boldsymbol{r}_f)$の固有値



K. (2005)

不安定方向に擾乱を加えた軌道

$\boldsymbol{r} = \boldsymbol{r}_f \pm \varepsilon ||\boldsymbol{r}_f|| \boldsymbol{e}_u$, $||\boldsymbol{r}_f|| = 0.31$, $\varepsilon = 10^{-4}$



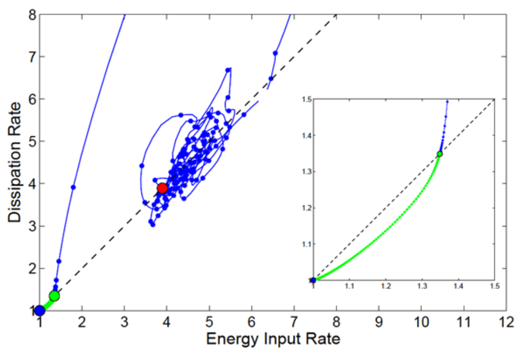
Poiseuille

Itano & Toh (2001)
Waleffe (2003)

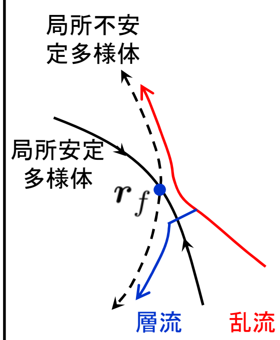
K. (2005)

定常解（Couette流）

Wang, Gibson & Waleffe (2007)



制御指針



- 状態点が不安定周期軌道に接近する際
- 制御（入力）パラメターを変化させ
- 状態点に乱流ー層流吸引域境界を横断させ
- 流れを層流化させる

制御（入力）パラメター σ

写像の線形化

写像　$r^{i+1} = P(r^i;\sigma)$

不動点　$r_f = P(r_f;0)$

$$r^{i+1} - r_f = \mathrm{D}_r P(r_f;0)(r^i - r_f) + \mathrm{D}_\sigma P(r_f;0)\sigma^i$$

制御入力の評価

ベクトル v_u

局所安定多様体上の任意のベクトル v_s に対して

$$v_u \cdot v_s = 0$$

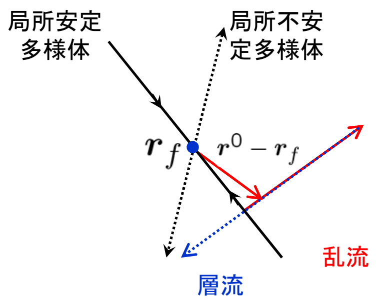
局所安定多様体に垂直な成分

$$v_u \cdot (r^{i+1} - r_f) = \lambda_u [v_u \cdot (r^i - r_f)] + \sigma^i [v_u \cdot \mathrm{D}_\sigma P(r_f;0)]$$

$$\sigma^0 = -c\lambda [v_u \cdot (r^0 - r_f)]/[v_u \cdot \mathrm{D}_\sigma P(r_f;0)]$$

$c > 1$　（OGY 法では $c = 1$）

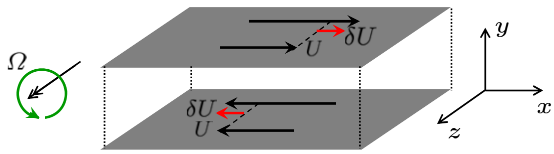
吸引域境界の横断



制御入力例

スパン方向軸まわりの回転

あるいは壁面速度の変化



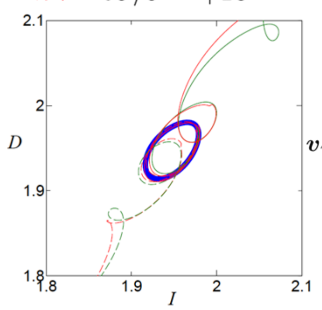
回転に伴う渦度　2Ω

壁面速度変化　δU

回転あるいは壁面速度変化

緑線　$2\Omega h/U = \pm 10^{-4}$

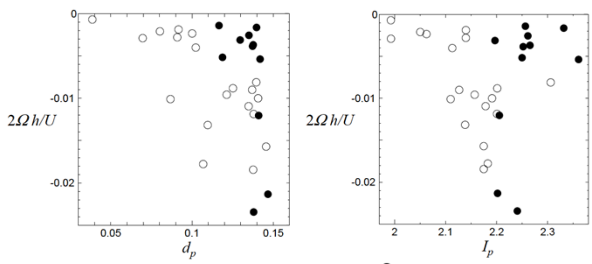
赤線　$\delta U/U = \mp 10^{-3}$



$v_u \cdot \mathrm{D}_\sigma P(r_f;0) \neq 0$

(回転による)層流化の試行結果

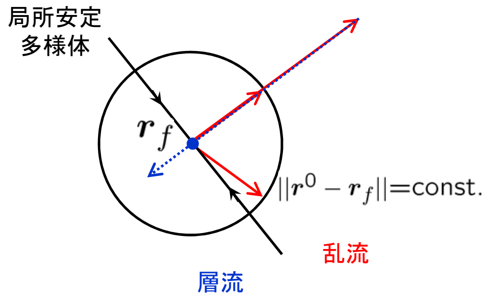
$c = 2$, $d_p < 0.15$,　○ 成功；● 失敗



OGY 法　$d_p < 10^{-2}$

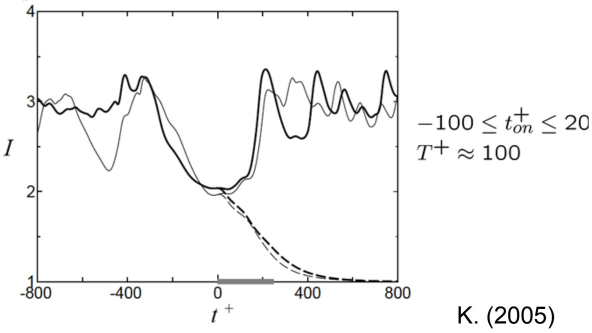
K. (2005)

制御パラメターの経験的決定



一定角速度，一定時間の回転

$I_p < 2.1 \rightarrow 2\Omega h/U = -3\times 10^{-3}$, $T^+ = 248$



K. (2005)

Pyragas の外力法による周期軌道の安定化

$(I_p < 2.2 \rightarrow T^+ \approx 4\times 10^4)$

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\underline{+k\mathcal{P}(u_{p\,i}-u_i)}$$

$u_{p\,i}$:周期解

$$\frac{\partial u_i}{\partial x_i} = 0$$

制御入力 $k\mathcal{P}(u_{p\,i}-u_i)$

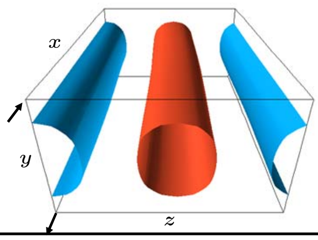
フーリエ・チェビシェフ係数 $\tilde{v}_{m,l,n}$, $\tilde{\omega}_{y\,m,l,n}$

$(m,n) = (0,\pm 1)$, $(\pm 1, 0)$, $(\pm 1,\pm 1)$, $(\pm 1,\mp 1)$

$l = 0$, 2

$kh/U = 0.1$

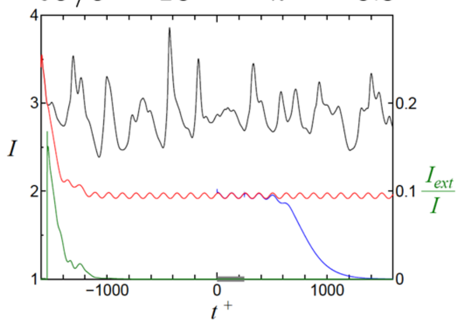
流れ方向速度の(青，赤)等値面 $\mathcal{P}u_{p\,x} = \pm 0.3U$



K. (2005)

安定化＋層流化

$\delta U/U = 10^{-3}$　　$k^+ = 0.9$



等方乱流と壁乱流の渦の動力学

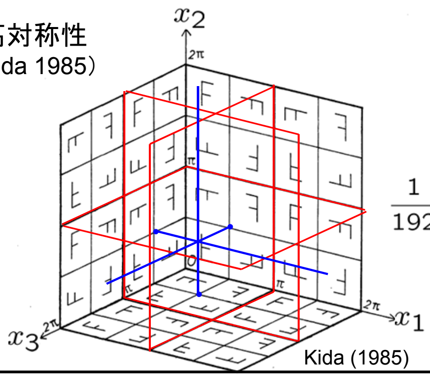
壁乱流 → 異方性ミニマル流 → 再生成サイクル（不安定周期運動）

等方乱流 → 高対称流 → 動力学？（不安定周期運動）

van Veen, Kida & K. (2006)

5.3 不安定周期軌道による等方乱流の記述
van Veen, Kida & K. (2006)

高対称性
（Kida 1985）



$\frac{1}{192}$

Kida (1985)

エネルギー，エンストロフィー，エネルギー散逸

$$\mathcal{E}(t) = \frac{1}{(2\pi)^3}\iiint \frac{1}{2}|\boldsymbol{u}|^2 \mathrm{d}V = \frac{1}{2}\sum|\tilde{\boldsymbol{u}}|^2$$

$$\mathcal{Q}(t) = \frac{1}{(2\pi)^3}\iiint \frac{1}{2}|\boldsymbol{\omega}|^2 \mathrm{d}V = \frac{1}{2}\sum|\tilde{\boldsymbol{\omega}}|^2$$

$$\epsilon(t) = 2\nu\mathcal{Q}(t)$$

エネルギー注入

注入波数 $k = \sqrt{11}$

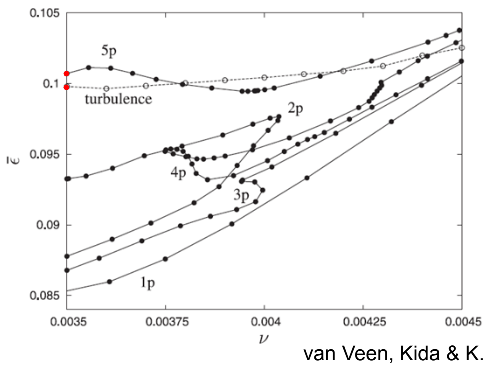
$$u_1 = \sin x_1(\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3)$$

$$u_2 = \sin x_2(\cos 3x_3 \cos x_1 - \cos x_3 \cos 3x_1)$$

$$u_3 = \sin x_3(\cos 3x_1 \cos x_2 - \cos x_1 \cos 3x_2)$$

$$e(t) = \epsilon + \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}$$

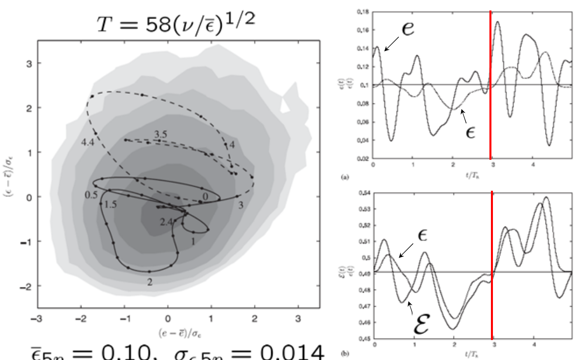
エネルギー散逸率



van Veen, Kida & K. (2006)

5周期解（$\nu = 0.0035$）

$T = 58(\nu/\bar{\epsilon})^{1/2}$



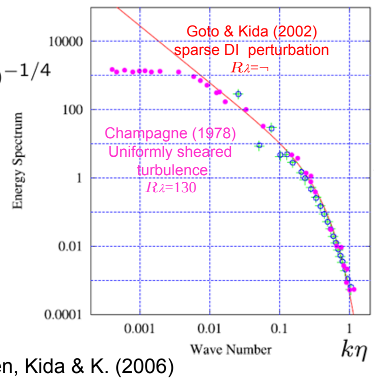
$\bar{\epsilon}_{5p} = 0.10$，$\sigma_{\epsilon\,5p} = 0.014$

$\bar{\epsilon} = 0.0998$，$\sigma_\epsilon = 0.0155$

van Veen, Kida & K. (2006)

エネルギースペクトル

$E_{||}(\bar{\epsilon}\nu^5)^{-1/4}$



○ 乱流運動

＋ 周期運動

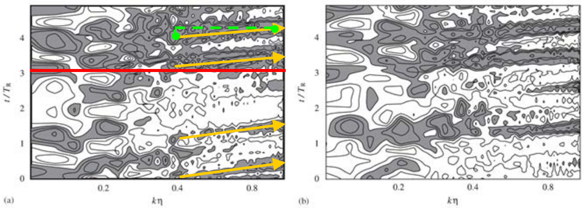
$$R_\lambda = \sqrt{\frac{10}{3}}\frac{\bar{\epsilon}}{\nu\sqrt{\mathcal{Q}}}$$
$$= 67$$

van Veen, Kida & K. (2006)

エネルギー伝達事象

$$E'(k,t) = [E(k,t) - E(k)]/\sigma_E(k)$$

周期運動 乱流運動

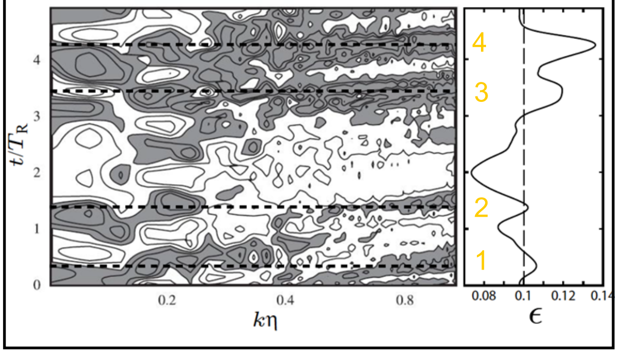


van Veen, Kida & K. (2006)

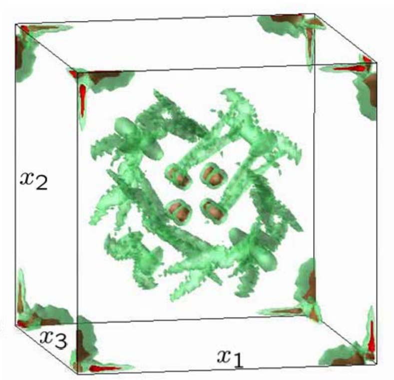
Kerr (1990); Kida & Ohkitani (1992)

エネルギー伝達と散逸

周期運動



渦構造の時間発展



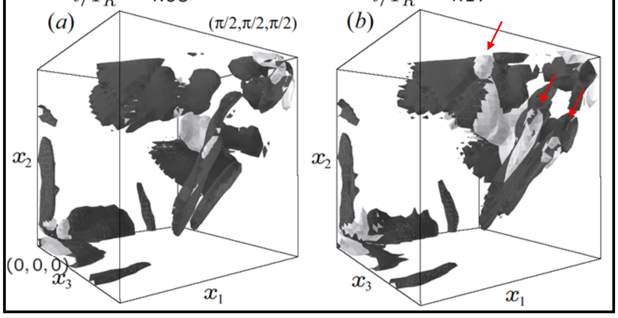
渦管の生成（エネルギー伝達事象4）

白等値面 $|\omega|^2 = 14\overline{Q}$, 灰色等値面 $e_{i,j}e_{i,j} = 8\overline{Q}$; $0 \le x_1,\ x_2,\ x_3 \le \pi/2$

$\pi/2/\eta \approx 60$

$\mathcal{E}$ 最大 $t/T_R = 4.10$

$t/T_R = 4.08$ $t/T_R = 4.17$



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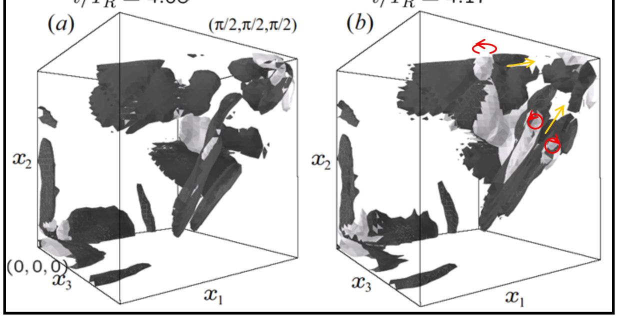
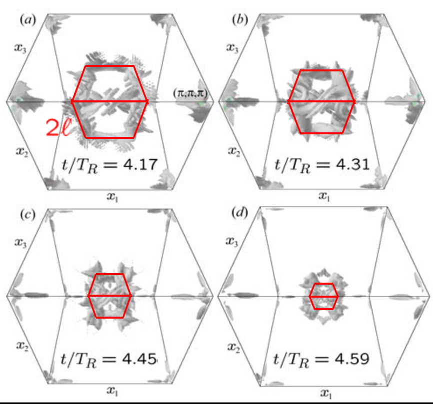


渦管の動力学(エネルギー伝達事象4)



ϵ 最大

$t/T_R = 4.25$

白等値面

$|\omega|^2 = 14\overline{Q}$

$0 \le x_1 \le \pi$

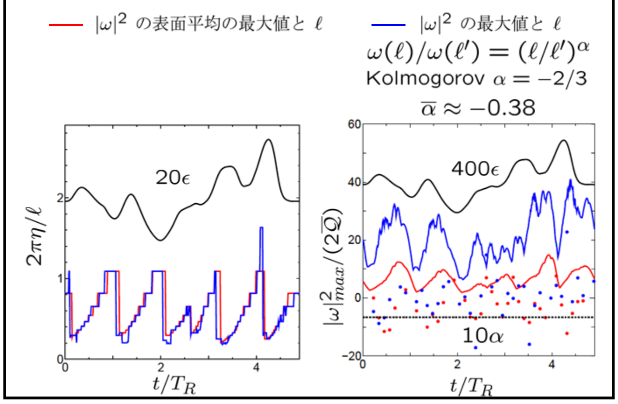
$0 \le x_2 \le \pi$

$0 \le x_3 \le \pi$

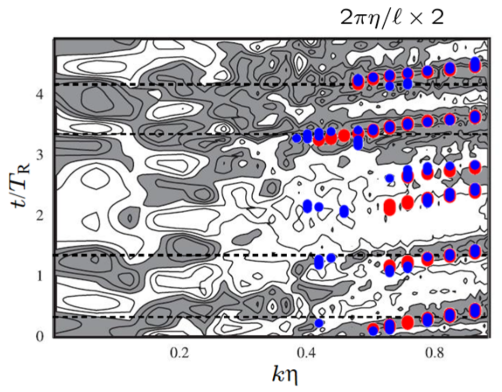
$\pi/\eta \approx 120$

Pelz (2001)

長さと渦度のスケールの時間発展



エネルギー伝達と渦構造のスケール



まとめ

- 乱流の統計法則
  Kolmogorovの4/5則
  Kolmogorovの相似則，Prandtlの壁法則
- 乱流の渦力学
  渦管の動力学：剪断流との相互作用
  渦層の動力学：巻き込み，不安定性
- 乱流の動力学と統計：不安定周期運動
  秩序構造の非線形動力学
  統計法則（Kolmogorov 則，壁法則）