Graduate School of Engineering Science, Osaka University

Turbulence Dynamics

Division of Nonlinear Mechanics

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0. Introduction
spatiotemporally chaotic motion in fluid

Dimotakis *et al*. (1981)

Higuchi (1993)

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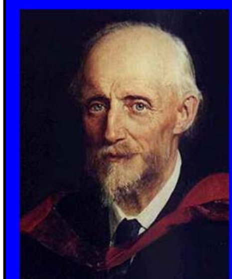
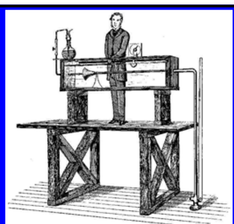
Transition to turbulence

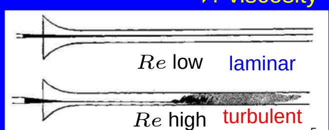


Osborne Reynolds
(1842 – 1912)



$$Re = \frac{Ud}{\nu}$$

U: velocity
d: diameter
ν: viscosity



Reynolds (1883)

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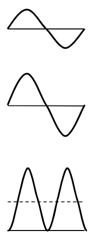
Turbulence phenomena
nonlinearity, dissipation, huge degrees of freedom

$$y = \sin x$$

$$ay + by = (a+b)\sin x$$

$$ay \times by = \frac{ab}{2} - \frac{ab}{2}\cos 2x$$



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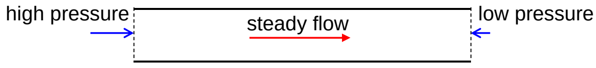
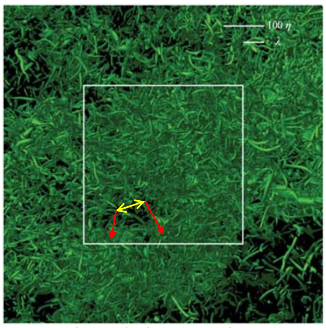
Turbulence phenomena
nonlinearity, dissipation, huge degrees of freedom



No energy conservation
pressure power = energy dissipation
Heat due to friction
kinetic energy ⇒ internal energy

Turbulence phenomena
nonlinearity, dissipation, huge degrees of freedom

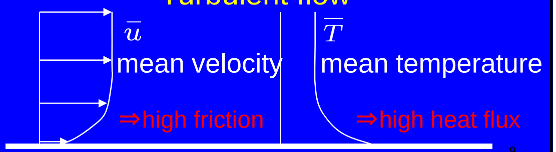


Ishihara, Kaneda, Yokokawa, Itakura & Uno (2007)

Momentum and heat transfer

Laminar flow

Turbulent flow

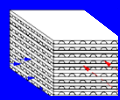


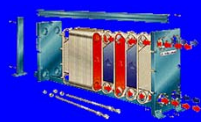
Engineering applications

Drag reduction

Airplane and pipeline

Heat transfer enhancement





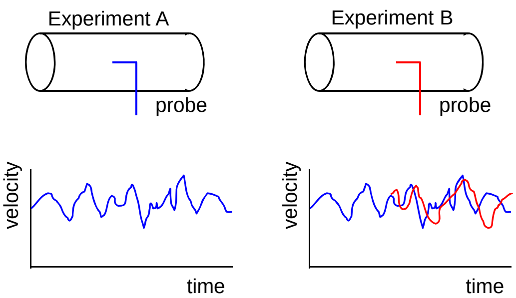
Compact and plate-shaped heat exchanger

Turbulence phenomena
nonlinearity, dissipation, huge degrees of freedom

Dynamical property
no reproducibility

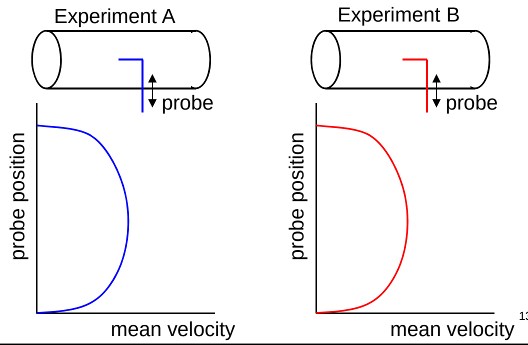
Statistical property
unclosed

Turbulence dynamics
no reproducibility of instantaneous velocity



chaos (sensitive dependence on initial condition)

Turbulence statistics
reproducibility of mean velocity



Closure problem

$$\frac{du}{dt} = -u^2, \quad u(t=0) = u_0$$

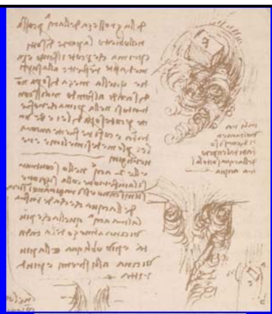
$$u = \frac{u_0}{1 + u_0 t}$$

$$\frac{d\overline{u}}{dt} = -\overline{u^2}, \quad \frac{d\overline{u^2}}{dt} = -2\overline{u^3}, \ \cdots$$

Davidson (2004)

Coherent structure

Leonardo da Vinci
(1452 – 1519)



- Statistics
  reproducible
  non-closed equation
- Dynamics, structures
  non-reproducible
  Navier-Stokes equation

Turbulence phenomena
nonlinearity, dissipation, huge degrees of freedom

Chaotic dynamics
no reproducibility

integration

Statistical law
unclosed

reproducibility

no assumption

Simple invariant sets
fixed point, periodic orbit

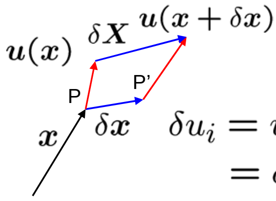
Wall turbulence: K. & Kida *JFM* 449 (2001)
Isotropic turbulence: van Veen, Kida & K. FDR 38 (2006)

References

- 「流体力学（前編）」
  今井　功，裳華房
- 「乱流力学」
  木田重雄，柳瀬眞一郎，朝倉書店
- 「乱流理論の基礎」
  後藤俊幸，朝倉書店
- Turbulence
  P. A. Davidson, Oxford University Press

1. Basic equations for imcompressible flow

1.1 vorticity



relative motion

2-point velocity difference

$\delta u_i \quad (i = 1, 2, 3)$

$$\delta u_i = u_i(\boldsymbol{x}+\delta\boldsymbol{x}) - u_i(\boldsymbol{x})$$
$$= \delta X_i - \delta x_i$$

$\delta x = |\delta \boldsymbol{x}|$

$$= \frac{\partial u_i}{\partial x_j}\delta x_j + O(\delta x^2)$$

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$\delta x \to 0$

$$\delta u_i = \frac{\partial u_i}{\partial x_j}\delta x_j$$

velocity-gradient tensor

$$D = \{d_{i,j}\} = \left\{\frac{\partial u_i}{\partial x_j}\right\}$$

eigen-polynomial of tensor T

$$F(\lambda) = -\det(T - \lambda I)$$
$$= \lambda^3 - \mathrm{tr}(T)\lambda^2 + Q(T)\lambda - \det(T)$$

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second invariant

$$Q(T) = \tfrac{1}{2}\left\{[\mathrm{tr}(T)]^2 - \mathrm{tr}(T^2)\right\}$$

imcompressible fluid

$$\mathrm{tr}(D) = d_{i,i} = \frac{\partial u_i}{\partial x_i} = \nabla \cdot \boldsymbol{u} = 0$$
$$Q(D) = \tfrac{1}{2}[(d_{i,i})^2 - d_{i,j}d_{j,i}]$$
$$= -\frac{1}{2}\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

$Q(D)$ is used for visualization of vortex

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decomposition in two parts

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{E = \{e_{i,j}\}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\Omega = \{\omega_{i,j}\}}$$

rate-of-strain tensor $\qquad$ vorticity tensor

$e_{i,j} = e_{j,i} \qquad \omega_{i,j} = -\omega_{j,i}$

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relative velocity represented by E

$$\delta \boldsymbol{u} = \tfrac{1}{2}E\delta\boldsymbol{x}$$

orthogonal matrix $A \quad (A^{-1} = A^t)$

$$\delta \boldsymbol{u} = A\delta\boldsymbol{u}' \qquad \delta\boldsymbol{x} = A\delta\boldsymbol{x}'$$
$$A\delta\boldsymbol{u}' = \tfrac{1}{2}EA\delta\boldsymbol{x}' \Rightarrow \delta\boldsymbol{u}' = \tfrac{1}{2}A^tEA\delta\boldsymbol{x}'$$

$$\tfrac{1}{2}A^tEA = \begin{pmatrix} \frac{\partial u'_1}{\partial x'_1} & & 0 \\ & \frac{\partial u'_2}{\partial x'_2} & \\ 0 & & \frac{\partial u'_3}{\partial x'_3} \end{pmatrix}$$

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incompressible fluid

$$\mathrm{tr}(E) = 2\mathrm{tr}(D) = 2\frac{\partial u_i}{\partial x_i} = 2\frac{\partial u'_i}{\partial x'_i} = 0$$
$$\frac{\partial u'_1}{\partial x'_1} \le \frac{\partial u'_2}{\partial x'_2} \le \frac{\partial u'_3}{\partial x'_3}$$
$$(1)\ \frac{\partial u'_1}{\partial x'_1} \le 0 \le \frac{\partial u'_2}{\partial x'_2} \le \frac{\partial u'_3}{\partial x'_3}$$
$$(2)\ \frac{\partial u'_1}{\partial x'_1} \le \frac{\partial u'_2}{\partial x'_2} \le 0 \le \frac{\partial u'_3}{\partial x'_3}$$

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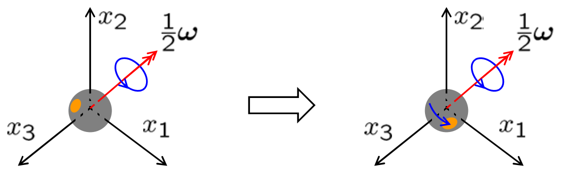
$$\delta u_i = \tfrac{1}{2}\omega_{i,j}\delta x_j$$
$$= \tfrac{1}{2}\epsilon_{j,i,k}\omega_k\delta x_j = \tfrac{1}{2}\epsilon_{i,k,j}\omega_k\delta x_j$$
$$\delta \boldsymbol{u} = \tfrac{1}{2}\Omega\delta\boldsymbol{x} = \tfrac{1}{2}\boldsymbol{\omega}\times\delta\boldsymbol{x}$$



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local relative fluid motion

$$\delta\boldsymbol{u} = D\delta\boldsymbol{x} = \tfrac{1}{2}E\delta\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{\omega}\times\delta\boldsymbol{x}$$

uniform strain in three orthogonal directions | uniform rotation

$$Q(D) = -\tfrac{1}{2}d_{i,j}d_{j,i}$$
$$= -\tfrac{1}{8}(e_{i,j}+\omega_{i,j})(e_{i,j}-\omega_{i,j})$$
$$= \tfrac{1}{8}(\|\Omega\|^2 - \|E\|^2)$$

$Q(D) > 0$ rotation $\quad Q(D) < 0$ strain

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1.2 vorticity equation

equation of continuity（mass conservation）

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{u} = 0$$
$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla$$

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stress tensor for incompressible Newtonian fluid

$$P = \{-p\delta_{i,j} + \mu e_{i,j}\}$$

Navier-Stokes equation（equation of motion）

$$\rho\frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial}{\partial x_j}[-p\delta_{i,j} + \mu e_{i,j}]$$
$$\frac{\mathrm{D}u_i}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}$$
$$\nu = \frac{\mu}{\rho}$$

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Invariance in Navier-Stokes equation

1．Translation in time
2．Translation in space
3．Rotation and reflection in space
4．Galilean transformation
5．Scale transformation

$$\boldsymbol{x}' = \lambda\boldsymbol{x},\quad \boldsymbol{u}' = \lambda^{\alpha/3}\boldsymbol{u},$$
$$t' = \lambda^{1-\alpha/3}t,\quad \rho' = \rho,$$
$$p' = \lambda^{2\alpha/3}p,\quad \nu' = \lambda^{1+\alpha/3}\nu$$

Navier-Stokes $\alpha = -3$

Euler α arbitrary $\quad$ Kolmogorov $\alpha = 1$

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divergence of Navier-Stokes equation

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_j\partial x_j} = -\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$$
$$-\left(\frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j} - \nu\frac{\partial^2}{\partial x_j\partial x_j}\right)\frac{\partial u_i}{\partial x_i}$$
$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_j\partial x_j} = -\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} = 2Q(D)$$
$$\nabla^2 p = 2\rho Q(D)$$

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non-locality of pressure

$$p(\boldsymbol{x}) = -\frac{\rho}{2\pi}\iiint_V \frac{Q(D')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}V' + \phi(\boldsymbol{x})$$

$$\nabla^2\phi = 0$$

rotation of Navier-Stokes equation

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2\right) + \boldsymbol{u}\times\boldsymbol{\omega} + \nu\nabla^2\boldsymbol{u}$$

$$\Longrightarrow \frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}) + \nu\nabla^2\boldsymbol{\omega}$$

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$$\frac{\mathrm{D}\omega_i}{\mathrm{D}t} = \underline{\omega_j\frac{\partial u_i}{\partial x_j}} + \nu\frac{\partial^2\omega_i}{\partial x_j\partial x_j}$$

vorticity-stretching term

$$\omega_j\frac{\partial u_i}{\partial x_j} = \omega_j\left(\tfrac{1}{2}e_{i,j} + \tfrac{1}{2}\omega_{i,j}\right)$$
$$= \tfrac{1}{2}e_{i,j}\omega_j + \tfrac{1}{2}\varepsilon_{i,k,j}\omega_k\omega_j$$

$$D\boldsymbol{\omega} = \tfrac{1}{2}E\boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\omega}\times\boldsymbol{\omega}$$
$$= \tfrac{1}{2}E\boldsymbol{\omega}$$

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vorticity equation

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \tfrac{1}{2}E\boldsymbol{\omega} + \nu\nabla^2\boldsymbol{\omega}$$

$$\frac{\mathrm{D}\omega_i}{\mathrm{D}t} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\omega_j + \nu\frac{\partial^2\omega_i}{\partial x_j\partial x_j}$$

$$\dot{\boldsymbol{\omega}} = \tfrac{1}{2}E\boldsymbol{\omega}$$

$$\dot{\boldsymbol{\omega}}' = \tfrac{1}{2}A^tEA\boldsymbol{\omega}'$$

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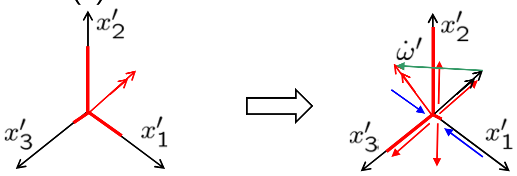
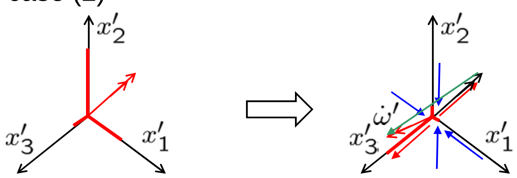
case (1)



case (2)



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equation for material line element

$$\frac{\mathrm{D}\delta\boldsymbol{\ell}}{\mathrm{D}t} = D\delta\boldsymbol{\ell}$$

$$\frac{\mathrm{D}\delta\boldsymbol{\ell}}{\mathrm{D}t} = \tfrac{1}{2}E\delta\boldsymbol{\ell} + \tfrac{1}{2}\boldsymbol{\omega}\times\delta\boldsymbol{\ell}$$

Helmholtz vortex theorem

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \tfrac{1}{2}E\boldsymbol{\omega}$$

$$\delta\boldsymbol{\ell} \parallel \boldsymbol{\omega} \rightarrow \boldsymbol{\omega} = a\delta\boldsymbol{\ell}$$

$$\frac{\mathrm{D}a}{\mathrm{D}t}\delta\boldsymbol{\ell} + a\frac{\mathrm{D}\delta\boldsymbol{\ell}}{\mathrm{D}t} = \tfrac{1}{2}aE\delta\boldsymbol{\ell} \rightarrow \frac{\mathrm{D}a}{\mathrm{D}t} = 0$$

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Time evolution of rate-of-strain tensor

$$\frac{\mathrm{D}E}{\mathrm{D}t} = -\tfrac{1}{2}(E^2 + \Omega^2) - \frac{2}{\rho}\Pi + \nu\nabla^2 E$$

Pressure Hessian

$$\Pi = \left\{\frac{\partial^2 p}{\partial x_i\partial x_j}\right\}$$

Time of evolution of vorticity tensor

$$\frac{\mathrm{D}\Omega}{\mathrm{D}t} = -\tfrac{1}{2}(E\Omega + \Omega E) + \nu\nabla^2\Omega$$

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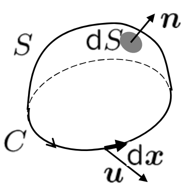
1.3 enstrophy dynamics
Kelvin's circulation theorem



$$\Gamma(C) = \oint_C \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{x} = \iint_S \boldsymbol{\omega}\cdot \boldsymbol{n}\mathrm{d}S$$

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \oint_C \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\cdot \mathrm{d}\boldsymbol{x} + \boldsymbol{u}\cdot\frac{\mathrm{D}\mathrm{d}\boldsymbol{x}}{\mathrm{D}t}\right)$$

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\boldsymbol{u} \qquad \frac{\mathrm{D}\mathrm{d}\boldsymbol{x}}{\mathrm{D}t} = (\mathrm{d}\boldsymbol{x}\cdot\nabla)\boldsymbol{u}$$

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \nu\oint_C (\nabla^2\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{x} \Rightarrow$$ conservation of Γ

$\nu = 0 \rightarrow \Gamma = \mathrm{const.}$

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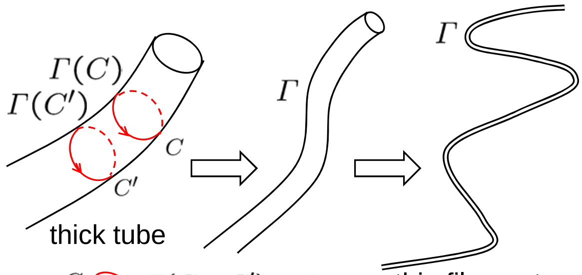
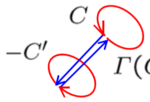
Motion of vortex tube





$\Gamma(C - C') = 0$

$\Gamma(C) + \Gamma(-C') = 0$

$\Gamma(C) - \Gamma(C') = 0$

$\Gamma = \omega\delta S$

$\delta S \rightarrow 0, \quad \omega \rightarrow \infty$

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Biot-Savart law

$$\boldsymbol{u} = \nabla\times\boldsymbol{A} + \nabla\phi \qquad \nabla\cdot\boldsymbol{A} = 0, \quad \nabla^2\phi = 0$$

$$\nabla^2\boldsymbol{A} = -\nabla\times\boldsymbol{u} = -\boldsymbol{\omega}$$

$$\boldsymbol{A} = \frac{1}{4\pi}\iiint_V \frac{\boldsymbol{\omega}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}V'$$

$$\nabla\times\boldsymbol{A} = \frac{1}{4\pi}\iiint_V \frac{\boldsymbol{\omega}(\boldsymbol{x}')\times(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\mathrm{d}V'$$

$$\approx \frac{\Gamma}{4\pi}\int_\ell \frac{\mathrm{d}\boldsymbol{\ell}'\times(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}$$

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Energy budget

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}|\boldsymbol{u}|^2\right) = -\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}|\boldsymbol{u}|^2 u_j + \frac{p}{\rho}u_j\right) + \nu u_i\frac{\partial^2 u_i}{\partial x_j\partial x_j}$$

$$u_i\frac{\partial^2 u_i}{\partial x_j\partial x_j} = \frac{\partial}{\partial x_j}(u_i d_{i,j}) - \|D\|^2$$
$$= \frac{\partial}{\partial x_j}(u_i e_{i,j}) - \tfrac{1}{2}\|E\|^2$$
$$= \frac{\partial}{\partial x_j}(u_i\omega_{i,j}) - |\boldsymbol{\omega}|^2$$
$$= \nabla\cdot(\boldsymbol{u}\times\boldsymbol{\omega}) - |\boldsymbol{\omega}|^2$$

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average $\overline{(\,)}^V = \frac{1}{V}\iiint_V (\,)\mathrm{d}V$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\overline{|\boldsymbol{u}|^2}^V\right) = \begin{cases} -\nu\overline{\|D\|^2}^V \\ -\nu\tfrac{1}{2}\overline{\|E\|^2}^V \\ -\nu\overline{|\boldsymbol{\omega}|^2}^V \end{cases} = -2\nu Q$$

energy dissipation

enstrophy $Q = \tfrac{1}{2}\overline{|\boldsymbol{\omega}|^2}^V$

$\nu = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\overline{|\boldsymbol{u}|^2}^V\right) = 0$ energy cascade

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Enstrophy budget

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}|\omega|^2\right) = -\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}|\omega|^2 u_j\right) + \frac{1}{2}\omega_i e_{i,j}\omega_j + \nu\omega_i\frac{\partial^2\omega_i}{\partial x_j\partial x_j}$$

$$\omega_i\frac{\partial^2\omega_i}{\partial x_j\partial x_j} = \nabla\cdot[\boldsymbol{\omega}\times(\nabla\times\boldsymbol{\omega})] - |\nabla\times\boldsymbol{\omega}|^2$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \tfrac{1}{2}\overline{\omega^t E\omega}^V + \frac{\nu}{V}\iint_{\partial V}\frac{\partial}{\partial n}\left(\tfrac{1}{2}|\omega|^2\right)\mathrm{d}S - \nu\overline{|\nabla\times\boldsymbol{\omega}|^2}^V$$

⇦ enstrophy variation by stretching

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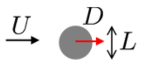
$\nu = 0 \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{1}{2}\overline{\boldsymbol{\omega}^t E \boldsymbol{\omega}}^V$ no conservation

2-d flow $\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$ enstrophy cascade

inviscid energy dissipation

$$\nu \to 0 \quad Q \to \infty \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\overline{|\boldsymbol{u}|^2}^V\right) = -2\nu Q < 0$$



$$C_D = \frac{D}{\frac{1}{2}\rho U^2 L^2}$$

$$\dot{W} = DU = \frac{1}{2}\rho U^3 L^2 C_D \qquad \epsilon \sim \frac{\dot{W}}{\rho L^3} \sim \frac{U^3}{L} C_D$$

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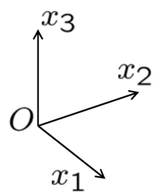
2. Dynamics of vortex layer and tube

2.1 stretched diffusing layer and tube

diffusing vortex layer



2-d flow

$$\boldsymbol{u} = u_1(x_1, x_2, t)\boldsymbol{e}_1 + u_2(x_1, x_2, t)\boldsymbol{e}_2$$

$$\boldsymbol{\omega} = \omega_3(x_1, x_2, t)\boldsymbol{e}_3 = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\boldsymbol{e}_3$$

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uni-directional flow $u_2 \equiv 0$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = \frac{\partial u_1}{\partial x_1} = 0$$

$$u_1 = u_1(x_2, t)$$

$$\omega_3 = -\frac{\partial u_1}{\partial x_2} = \omega_3(x_2, t)$$

$$\frac{\partial \omega_3}{\partial t} = \nu \frac{\partial^2 \omega_3}{\partial x_2^2}$$

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similarity solution $\omega_3(t = 0) = 2U_0\delta(x_2)$

$$\omega_3(x_2, t) \propto \frac{U_0}{\sqrt{\nu t}} f\left(\underbrace{\frac{x_2}{\sqrt{\nu t}}}_{=\xi}\right)$$

similarity variable

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{2}\xi f\right) = 0$$

symmetric solution $f(\xi) \propto \mathrm{e}^{-\xi^2/4}$

$$\Rightarrow \quad \omega_3 = \frac{1}{\sqrt{\pi}} \frac{U_0}{\sqrt{\nu t}} \exp\left(-\frac{x_2^2}{4\nu t}\right)$$

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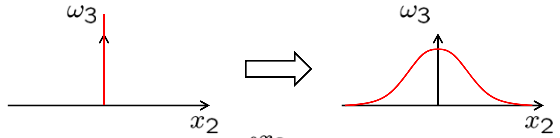

velocity
$$u_1 = -\int_0^{x_2} \omega_3(x_2', t)\mathrm{d}x_2' = -\frac{2U_0}{\sqrt{\pi}}\mathrm{erf}\left(\frac{x_2}{2\sqrt{\nu t}}\right)$$

$$\mathrm{erf}(x) = \int_0^x \mathrm{e}^{-s^2}\mathrm{d}s \qquad \mathrm{erf}(\pm\infty) = \pm\frac{\sqrt{\pi}}{2}$$

circulation density
$$\int_{-\infty}^{+\infty} \omega_3 \mathrm{d}x_2 = -[u_1]_{-\infty}^{+\infty} = 2U_0$$

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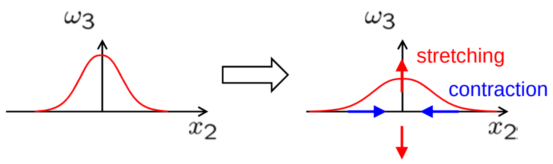
stretched and diffusing layer



2-d incompressible straining flow

$$\boldsymbol{u} = u_1(x_2, t)\boldsymbol{e}_1 - \gamma x_2 \boldsymbol{e}_2 + \gamma x_3 \boldsymbol{e}_3$$

$$\boldsymbol{\omega} = \omega_3(x_2, t)\boldsymbol{e}_3 = -\frac{\partial u_1}{\partial x_2}\boldsymbol{e}_3$$

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Steady solution (Burgers layer) $\omega_3(x_2)$

$$\frac{\partial \omega_3}{\partial t} - \gamma x_2 \frac{\partial \omega_3}{\partial x_2} = \gamma \omega_3 + \nu \frac{\partial^2 \omega_3}{\partial x_2^2}$$

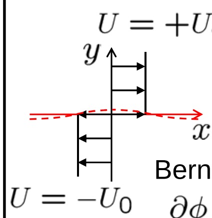
$$\nu \frac{\mathrm{d}^2 \omega_3}{\mathrm{d} x_2^2} + \gamma \frac{\mathrm{d}}{\mathrm{d} x_2}(x_2 \omega_3) = 0$$

symmetric solution

$$\omega_3 = \sqrt{\frac{2}{\pi}} \frac{U_0}{\sqrt{\nu/\gamma}} \exp\left(-\frac{x_2^2}{2\nu/\gamma}\right)$$

$$u_1 = -\frac{2U_0}{\sqrt{\pi}} \mathrm{erf}\left(\frac{x_2}{\sqrt{2\nu/\gamma}}\right)$$

Kelvin-Helmholtz instability



infinitesimally thin layer

$$\boldsymbol{u} = U \boldsymbol{e}_x + \nabla \phi$$

$$\nabla^2 \phi = 0$$

Bernoulli equation $\rho = 1$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2 + (P + p) = f(t)$$

Linearized equation

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \phi + p = 0$$

Material surface $S(x, y) + s(x, y, t) = 0$

$$\frac{\mathrm{D}}{\mathrm{D}t}(S + s) = 0$$

Linearized equation $S(x, y) \equiv y = 0$

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) s + \frac{\partial \phi}{\partial y} \frac{\partial S}{\partial y} = 0$$

normal mode $f = \phi, \ p, \ s$

$$f(x, y, t) = \mathrm{Re}\left[\tilde{f}(y)\, \mathrm{e}^{\mathrm{i}\alpha(x - ct)}\right]$$

perturbation equation

$$\frac{\mathrm{d}^2 \tilde{\phi}}{\mathrm{d} y^2} - \alpha^2 \tilde{\phi} = 0$$

$$\mathrm{i}\alpha(U - c)\tilde{\phi} + \tilde{p} = 0$$

$$\mathrm{i}\alpha(U - c)\tilde{s} + \frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}y} = 0 \quad (y = 0)$$

$$\tilde{\phi}(y) = \begin{cases} \tilde{\phi}^+(y) & (y > 0) \\ \tilde{\phi}^-(y) & (y < 0) \end{cases}$$

Matching conditions $(y = 0)$

$$(U_0 - c)\tilde{\phi}^+ = (-U_0 - c)\tilde{\phi}^-$$

$$(-U_0 - c)\frac{\mathrm{d}\tilde{\phi}^+}{\mathrm{d}y} = (U_0 - c)\frac{\mathrm{d}\tilde{\phi}^-}{\mathrm{d}y}$$

eigensolution

$$\tilde{\phi}^+ = A\mathrm{e}^{-\alpha y} \qquad \tilde{\phi}^- = B\mathrm{e}^{+\alpha y}$$

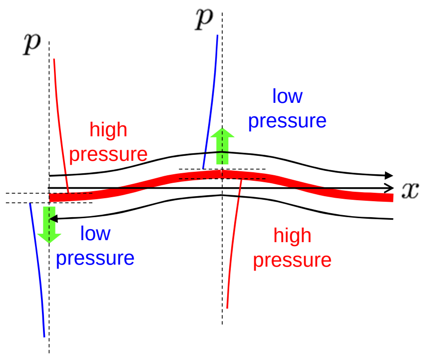
$$(U_0 - c)A + (U_0 + c)B = 0$$

$$(U_0 + c)A - (U_0 - c)B = 0$$

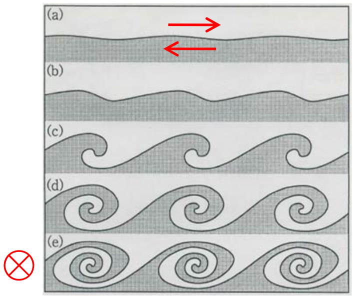
$$c/U_0 = \pm \mathrm{i} \Longrightarrow \alpha \mathrm{Im}(c) = \pm \alpha U_0$$

$$B/A = \pm \mathrm{i}$$

Instability mechanism



nonlinear rolling-up of vortex sheet



Krasny (1986)

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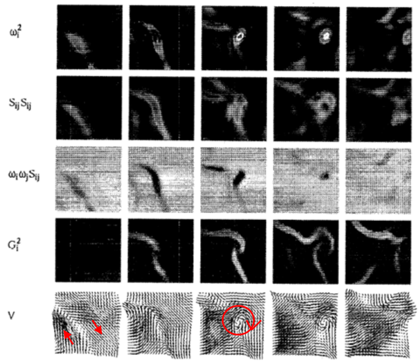
layer-tube transition



Ruetsch & Maxey (1992)

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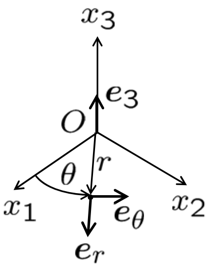
diffusing vortex tube



2-d flow

$$\boldsymbol{u} = u_r(r,\theta,t)\boldsymbol{e}_r + u_\theta(r,\theta,t)\boldsymbol{e}_\theta$$

$$\boldsymbol{\omega} = \omega_3(x_1,x_2,t)\boldsymbol{e}_3$$

$$\frac{\partial \boldsymbol{e}_r}{\partial r} = 0 \qquad \frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta$$

$$\frac{\partial \boldsymbol{e}_\theta}{\partial r} = 0 \qquad \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r$$

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Circular flow $u_r \equiv 0$

$$\nabla_\perp \cdot \boldsymbol{u} = \left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\right)\cdot(u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta)$$

$$= \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0$$

$$\frac{\partial u_\theta}{\partial \theta} = 0$$

$$u_\theta = u_\theta(r,t)$$

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$$\boldsymbol{\omega} = \nabla_\perp \times \boldsymbol{u}$$

$$= \left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\right)\times(u_\theta\boldsymbol{e}_\theta)$$

$$= \frac{\partial u_\theta}{\partial r}\boldsymbol{e}_r\times\boldsymbol{e}_\theta - \frac{u_\theta}{r}\boldsymbol{e}_\theta\times\boldsymbol{e}_r$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\boldsymbol{e}_3$$

$$\omega_3 = \omega_3(r,t) = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)$$

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$$\boldsymbol{u}\cdot\nabla_\perp = (u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta)\cdot\left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

$$= u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}$$

$$\nabla_\perp^2 = \left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\right)\cdot\left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

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2-d flow

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla_\perp)\boldsymbol{\omega} = \nu \nabla_\perp^2 \boldsymbol{\omega}$$

$$\frac{\partial \omega_3}{\partial t} + \left(u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} \right) \omega_3$$

$$= \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) \omega_3$$

circular flow

$$\frac{\partial \omega_3}{\partial t} = \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) \omega_3$$

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similarity solution $\omega_3(t=0) = \Gamma \delta(r)/(\pi r)$

$$\omega_3(r,t) \propto \frac{\Gamma}{\nu t} f\left(\underbrace{\frac{r}{\sqrt{\nu t}}}_{=\,\xi} \right)$$

similarity variable

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi \frac{\mathrm{d}f}{\mathrm{d}\xi} \right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\tfrac{1}{2}\xi^2 f \right) = 0$$

regular solution $f(\xi) \propto \mathrm{e}^{-\xi^2/4}$

$$\Rightarrow \quad \omega_3 = \frac{\Gamma}{4\pi \nu t} \exp\left(-\frac{r^2}{4\nu t} \right)$$

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velocity

$$u_\theta = \frac{1}{r}\int_0^r r' \omega_3(r',t)\mathrm{d}r'$$

$$= \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t} \right) \right]$$

circulation on circle of radius r

$$\Gamma(r;t) = \int_0^{2\pi} u_\theta r \mathrm{d}\theta = \Gamma \left[1 - \exp\left(-\frac{r^2}{4\nu t} \right) \right]$$

$$\Gamma(r=\infty;t) = \Gamma$$

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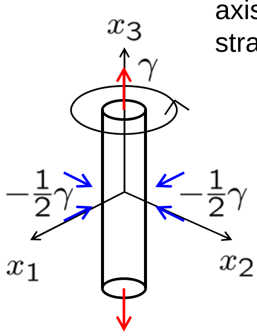
stretched diffusing vortex tube



axisymmetric incompressible straining flow

$$\boldsymbol{u} = u_\theta(r,t)\boldsymbol{e}_\theta$$

$$-\frac{\gamma}{2} r \boldsymbol{e}_r + \gamma x_3 \boldsymbol{e}_3$$

$$\boldsymbol{\omega} = \omega_3(r,t)\boldsymbol{e}_3$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\boldsymbol{e}_3$$

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unsteady solution $\omega_3(r,t)$

$$\frac{\partial \omega_3}{\partial t} - \frac{\gamma}{2} r \frac{\partial \omega_3}{\partial r} = \gamma \omega_3 + \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) \omega_3$$

Lundgren (1982) transformation

stretch parameter $S(t)$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \gamma S \qquad S(t=0) = 1$$

$$S = \exp\left(\int_0^t \gamma(t')\mathrm{d}t' \right) = \mathrm{e}^{\gamma t}$$

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$$R = [S(t)]^{1/2} r = \mathrm{e}^{\gamma t/2} r$$

$$T = T_0 + \int_0^t S(t')\mathrm{d}t' = T_0 + (\mathrm{e}^{\gamma t} - 1)/\gamma$$

$$\Omega(R,T) = \omega_3(r,t)/S(t) = \omega_3 \mathrm{e}^{-\gamma t}$$

$$\frac{\partial}{\partial t} = \frac{\partial T}{\partial t}\frac{\partial}{\partial T} + \frac{\partial R}{\partial t}\frac{\partial}{\partial R} = S\frac{\partial}{\partial T} + \frac{\gamma}{2} R \frac{\partial}{\partial R}$$

$$\frac{\partial}{\partial r} = \frac{\partial T}{\partial r}\frac{\partial}{\partial T} + \frac{\partial R}{\partial r}\frac{\partial}{\partial R} = S^{1/2}\frac{\partial}{\partial R}$$

$$\frac{\partial^2}{\partial r^2} = S\frac{\partial^2}{\partial R^2}$$

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vorticity equation

$$\frac{\partial}{\partial t}(S\Omega) - \frac{\gamma}{2} r \frac{\partial}{\partial r}(S\Omega) = \gamma S\Omega + \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) S\Omega$$

$$\gamma S\Omega + S\left(S\frac{\partial}{\partial T} + \frac{\gamma}{2} R \frac{\partial}{\partial R}\right)\Omega - \frac{\gamma}{2} S R \frac{\partial \Omega}{\partial R} = \gamma S\Omega + \nu S^2\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R}\right)\Omega$$

$$\frac{\partial \Omega}{\partial T} = \nu\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R\partial R}\right)\Omega$$

Lamb-Oseen vortex

$$\Omega = \frac{\Gamma}{4\pi\nu T}\exp\left(-\frac{R^2}{4\nu T}\right)$$

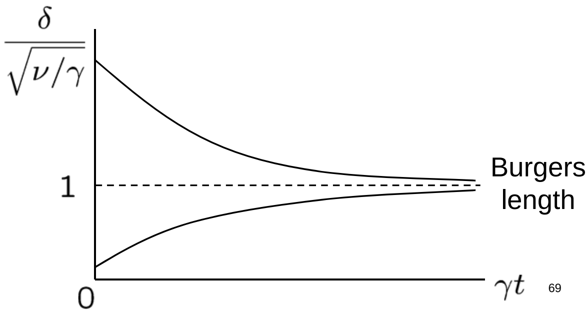
stretched diffusing vortex tube

$$\omega_3 = S\Omega = \frac{\Gamma}{4\pi\delta^2}\exp\left(-\frac{r^2}{4\delta^2}\right)$$

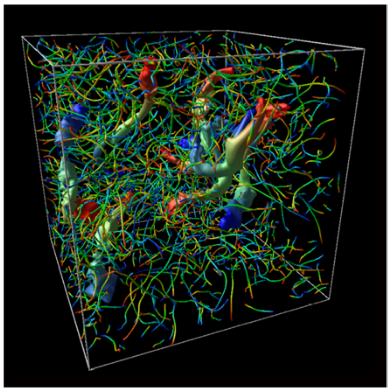
$$u_\theta = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\delta^2}\right)\right]$$

thickness of vortex tube

$$\delta(t) = \sqrt{\nu/\gamma + (\nu T_0 - \nu/\gamma)e^{-\gamma t}}$$



Coherent structures in isotropic turbulence



Vortex tube

diameter

$10(\nu^3/\bar{\epsilon})^{1/4}$

velocity

$3(\nu\bar{\epsilon})^{1/4}$

energy dissipation

$\bar{\epsilon} = \nu\|E\|^2/2$

Kida & Miura (2000)

Vortex tubes in small scale turbulence

Burgers length $\sqrt{\nu/\gamma} \sim (\nu^3/\bar{\epsilon})^{1/4}$

strain time scale $\frac{1}{\gamma} \sim \sqrt{\nu/\bar{\epsilon}}$

Burgers velocity $\frac{\Gamma}{\sqrt{\nu/\gamma}} \sim (\nu\bar{\epsilon})^{1/4}$

Vortex Reynolds number $\frac{\Gamma}{\nu} \sim 1$

turnover time $\frac{\nu}{\Gamma}\frac{1}{\gamma} \sim \sqrt{\nu/\bar{\epsilon}}$

Coherent structures in near-wall turbulence

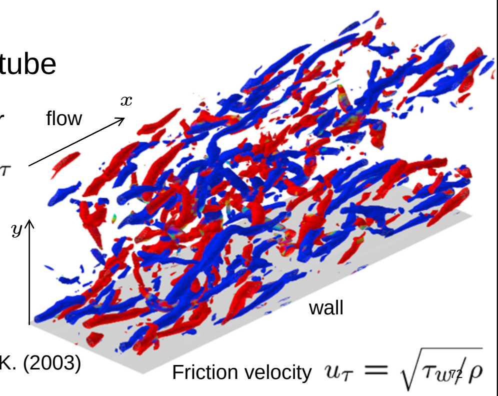
Vortex tube

diameter

$30\nu/u_\tau$

velocity

u_τ



K. (2003)

Friction velocity $u_\tau = \sqrt{\tau_w/\rho}$

Streamwise vortex in near-wall turbulence

Burgers length $\sqrt{\nu/\gamma} \sim \nu/u_\tau$

strain time scale $\dfrac{1}{\gamma} \sim \nu/u_\tau^2$

Burgers velocity $\dfrac{\Gamma}{\sqrt{\nu/\gamma}} \sim u_\tau$

Vortex Reynolds number $\dfrac{\Gamma}{\nu} \sim 1$

turnover time $\dfrac{\nu}{\Gamma}\dfrac{1}{\gamma} \sim \nu/u_\tau^2$

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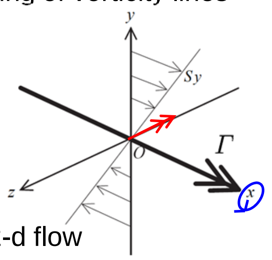
2.2 Wrapping of vorticity lines



generalized 2-d flow

vorticity

$$\boldsymbol{\omega} = \omega_x(y,z,t)\boldsymbol{e}_x + \omega_y(y,z,t)\boldsymbol{e}_y + \omega_z(y,z,t)\boldsymbol{e}_z$$

velocity

$$\boldsymbol{u} = u(y,z,t)\boldsymbol{e}_x + v(y,z,t)\boldsymbol{e}_y + w(y,z,t)\boldsymbol{e}_z$$

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streamfunction

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \quad v = \frac{\partial \psi}{\partial z} \qquad w = -\frac{\partial \psi}{\partial y}$$

$$\omega_x = -\nabla_\perp^2 \psi = -\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi$$

$$\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}$$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y}$$

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vorticity

$$\boldsymbol{\omega}(y,z,t) = -\nabla_\perp^2 \psi\, \boldsymbol{e}_x + \frac{\partial u}{\partial z}\boldsymbol{e}_y - \frac{\partial u}{\partial y}\boldsymbol{e}_z$$

velocity

$$\boldsymbol{u}(y,z,t) = u\, \boldsymbol{e}_x + \frac{\partial \psi}{\partial z}\boldsymbol{e}_y - \frac{\partial \psi}{\partial y}\boldsymbol{e}_z$$

projection of vorticity line on (y,z)-plane

$$\left(\frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)(\omega_y, \omega_z)^t = \left(\frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)\left(\frac{\partial u}{\partial z}, -\frac{\partial u}{\partial y}\right)^t = 0$$

$\nabla_\perp u \perp (\omega_y, \omega_z) \Rightarrow$ $u =$ const. projection of vorticity line

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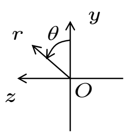
vorticity equation

$$\frac{\partial \omega_x}{\partial t} + v\frac{\partial \omega_x}{\partial y} + w\frac{\partial \omega_x}{\partial z} = \nu \nabla_\perp^2 \omega_x$$

initial condition : $\omega_x(t=0) = \dfrac{\Gamma \delta(r)}{\pi r}$



diffusing tube $(\xi = \frac{1}{2} r(\nu t)^{-1/2})$

$$\omega_x(r,t) = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) = \frac{\Gamma}{4\pi\nu t}\mathrm{e}^{-\xi^2}$$

$$u_\theta(r,t) = \frac{\Gamma}{2\pi r}\left(1 - \mathrm{e}^{-\xi^2}\right)$$

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Navier-Stokes equation

$$\frac{\partial u}{\partial t} + \frac{u_\theta}{r}\frac{\partial u}{\partial \theta} = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)u$$

initial condition : $u(t=0) = Sy = Sr\cos\theta$

similarity solution $u(r,\theta,t) = Sr\,\mathrm{Re}[f(\xi)\,\mathrm{e}^{-\mathrm{i}\theta}]$

$$f'' + \left(2\xi + \frac{3}{\xi}\right)f' + \mathrm{i}\frac{\Gamma}{2\pi\nu}\frac{1 - \mathrm{e}^{-\xi^2}}{\xi^2}f = 0$$

boundary condition : $f(\xi = \infty) = 1$

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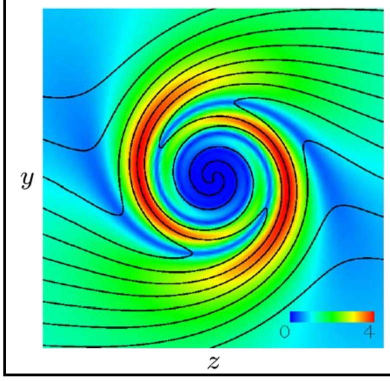
Pearson-Abernathy (1984); Moore (1985)
spiral vortex layers



cross-axial vorticity

$$\frac{\sqrt{\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2}}{S}$$

$$\frac{\Gamma}{2\pi\nu}=100$$

$40\sqrt{\nu t}$

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3. Universal statistical laws in isotropic and near-wall turbulence

3.1 Kolmogorov 4/5 law

ensemble average $\overline{x}=\int_{-\infty}^{\infty} x f(x)\mathrm{d}x$

$$\overline{x}=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}x_i$$

temporal average $\overline{x}^T=\lim_{T\to\infty}\frac{1}{T}\int_0^T x\mathrm{d}t$

spatial average $\overline{x}^V=\lim_{V\to\infty}\frac{1}{V}\int_V x\mathrm{d}V$

ergodicity

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Incompressible Navier-Stokes equation $(\rho=1)$

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}$$

$$\frac{\partial u_i}{\partial x_i}=0$$

Reynolds decomposition

$$u_i=\overline{u}_i+u'_i,\quad p=\overline{p}+p'$$

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Average equation

$$\frac{\partial \overline{u}_i}{\partial t}+\overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j}=-\frac{\partial \overline{p}}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\nu\left(\frac{\partial \overline{u}_i}{\partial x_j}+\frac{\partial \overline{u}_j}{\partial x_i}\right)-\overline{u'_i u'_j}\right]$$

Reynolds stress

$$\frac{\partial \overline{u}_i}{\partial x_i}=0$$

Fluctuation equation

$$\frac{\partial u'_i}{\partial t}+\left(\overline{u}_j+u'_j\right)\frac{\partial u'_i}{\partial x_j}+u'_j\frac{\partial \overline{u}_i}{\partial x_j}-\frac{\partial}{\partial x_j}\overline{u'_i u'_j}=-\frac{\partial p'}{\partial x_i}+\nu\frac{\partial^2 u'_i}{\partial x_j\partial x_j}$$

$$\frac{\partial u'_i}{\partial x_i}=0$$

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Energy equation

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}|\boldsymbol{u}'|^2\right)+\frac{\partial}{\partial x_j}\left[\left(\overline{u}_j+u'_j\right)\tfrac{1}{2}|\boldsymbol{u}'|^2\right]$$

$$+u'_i u'_j\frac{\partial \overline{u}_i}{\partial x_j}-u'_i\frac{\partial}{\partial x_j}\overline{u'_i u'_j}=-\frac{\partial}{\partial x_i}\left(p'u'_i\right)+\nu u'_i\frac{\partial^2 u'_i}{\partial x_j\partial x_j}$$

Turbulence energy equation

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\overline{|\boldsymbol{u}'|^2}\right)=-\frac{\partial}{\partial x_j}\left[\overline{\left(\overline{u}_j+u'_j\right)\tfrac{1}{2}|\boldsymbol{u}'|^2}-\overline{\left(-p'\delta_{i,j}+\nu e'_{i,j}\right)u'_i}\right]$$

$$-\frac{\nu}{2}\overline{e'_{i,j}e'_{i,j}}-\overline{u'_i u'_j}\frac{\partial \overline{u}_i}{\partial x_j}$$

$$e'_{i,j}=\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}$$

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Energy dissipation

flow → molecule: $\frac{\nu}{2}e_{i,j}e_{i,j}=\nu e_{i,j}\frac{\partial u_i}{\partial x_j}$

Turbulence energy production

mean flow ↔ eddy: $-\overline{u'_i u'_j}\frac{\partial \overline{u}_i}{\partial x_j}$

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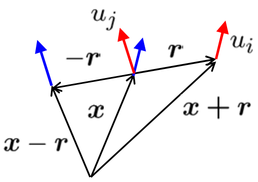
2nd-order velocity correlation tensor



$$
\begin{aligned}
R_{i,j}(\boldsymbol{r}) &= \overline{u_i(\boldsymbol{x}+\boldsymbol{r})u_j(\boldsymbol{x})} \\
&= \overline{u_i(\boldsymbol{x})u_j(\boldsymbol{x}-\boldsymbol{r})} \\
&= \overline{u_i(\boldsymbol{x})u_j(\boldsymbol{x}')} \\
&= R_{i,j}(\boldsymbol{x},\boldsymbol{x}')
\end{aligned}
$$

$$\boldsymbol{x}' = \boldsymbol{x}-\boldsymbol{r}$$

$$\overline{u_\alpha(\boldsymbol{x}+\boldsymbol{r})u_\beta(\boldsymbol{x})} = R_{i,j}\alpha_i\beta_j$$

$$\frac{\partial}{\partial x_k}R_{i,j}(\boldsymbol{x},\boldsymbol{x}') = \frac{\partial}{\partial r_k}R_{i,j}(\boldsymbol{r}) = -\frac{\partial}{\partial x'_k}R_{i,j}(\boldsymbol{x},\boldsymbol{x}')$$

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incompressibility

$$
\begin{aligned}
\frac{\partial}{\partial r_i}R_{i,j}(\boldsymbol{r}) &= \frac{\partial}{\partial x_i}R_{i,j}(\boldsymbol{x},\boldsymbol{x}') \\
&= \overline{\frac{\partial u_i}{\partial x_i}(\boldsymbol{x})u_j(\boldsymbol{x}')} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial r_j}R_{i,j}(\boldsymbol{r}) &= -\frac{\partial}{\partial x'_j}R_{i,j}(\boldsymbol{x},\boldsymbol{x}') \\
&= -\overline{u_i(\boldsymbol{x})\frac{\partial u_j}{\partial x_j}(\boldsymbol{x}')} \\
&= 0
\end{aligned}
$$

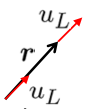
86

2nd-order velocity correlation tensor

$$R_{i,j}(\boldsymbol{r}) = F(r)\frac{r_i r_j}{r^2} + G(r)\delta_{i,j}$$

$$R_{i,j}(\boldsymbol{r}) = R_{i,j}(-\boldsymbol{r}) = R_{j,i}(\boldsymbol{r})$$

RMS（root-mean-square）velocity



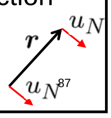
$$U^2 = \tfrac{1}{3}R_{i,i}(0) = \tfrac{1}{3}\overline{|\boldsymbol{u}|^2}$$

2nd-order longitudinal velocity correlation function

$$f(r) = \overline{u_L(\boldsymbol{x}+\boldsymbol{r})u_L(\boldsymbol{x})}/U^2$$

2nd-order lateral velocity correlation function

$$g(r) = \overline{u_N(\boldsymbol{x}+\boldsymbol{r})u_N(\boldsymbol{x})}/U^2$$



87

Longitudinal correlation

$$U^2 f(r) = R_{i,j}(\boldsymbol{r})\frac{r_i r_j}{r^2} = F(r) + G(r)$$

Lateral correlation

$$\boldsymbol{u}_N = \boldsymbol{u} - \frac{\boldsymbol{u}\cdot\boldsymbol{r}}{r}\frac{\boldsymbol{r}}{r}$$

$$u_{Nk} = \left(\delta_{k,i} - \frac{r_k r_i}{r^2}\right)u_i$$

$$
\begin{aligned}
U^2 g(r) &= \tfrac{1}{2}\left(\delta_{k,i} - \frac{r_k r_i}{r^2}\right)\left(\delta_{k,j} - \frac{r_k r_j}{r^2}\right)\overline{u_i(\boldsymbol{x}+\boldsymbol{r})u_j(\boldsymbol{x})} \\
&= \tfrac{1}{2}\left(\delta_{i,j} - \frac{r_i r_j}{r^2}\right)R_{i,j}(\boldsymbol{r}) = G(r)
\end{aligned}
$$

$$F(r) = U^2[f(r) - g(r)] \qquad G(r) = U^2 g(r)$$

88

Incompressibility

$$
\begin{aligned}
\frac{\partial}{\partial r_i}R_{i,j}(\boldsymbol{r}) &= \frac{\partial}{\partial x_i}R_{i,j}(\boldsymbol{x},\boldsymbol{x}') \\
&= U^2\left[\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{2}{r}(f-g)\right]\frac{r_j}{r} \\
&= 0
\end{aligned}
$$

$$\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{2}{r}(f-g) = 0$$

$$g = f + \frac{r}{2}\frac{\mathrm{d}f}{\mathrm{d}r}$$

89

Homogeneity

$$R_{1,1}(r_1\boldsymbol{e}_1) = \overline{u_1(\boldsymbol{x}+r_1\boldsymbol{e}_1)u_1(\boldsymbol{x})}$$

$$\frac{\partial}{\partial x_1}R_{1,1}(\boldsymbol{x},\boldsymbol{x}') = -\frac{\partial}{\partial x'_1}R_{1,1}(\boldsymbol{x},\boldsymbol{x}')$$

$$\frac{\partial}{\partial x_1}R_{1,1}(\boldsymbol{x},\boldsymbol{x}) = -\frac{\partial}{\partial x_1}R_{1,1}(\boldsymbol{x},\boldsymbol{x}) = 0$$

Taylor length

$$\lambda = \left(-\left.\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}\right|_{r=0}\right)^{-1/2}$$

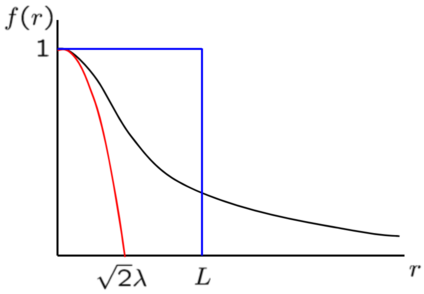
$$f(r) = 1 - \tfrac{1}{2}\left(\frac{r}{\lambda}\right)^2 + \cdots$$

$$g(r) = 1 - \left(\frac{r}{\lambda}\right)^2 + \cdots$$

90

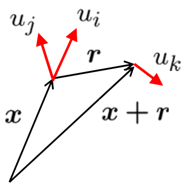
Integral length

$$L = \int_0^\infty f(r)\mathrm{d}r \qquad \int_0^\infty g(r)\mathrm{d}r = \tfrac{1}{2}L$$



91

3rd-order velocity correlation tensor



$$R_{i,j,k}(\boldsymbol{r}) = \overline{u_i(\boldsymbol{x})u_j(\boldsymbol{x})u_k(\boldsymbol{x}+\boldsymbol{r})}$$

$$R_{i,j,k}(\boldsymbol{r}) = R_{j,i,k}(\boldsymbol{r})$$

$$\frac{\partial}{\partial r_k}R_{i,j,k}(\boldsymbol{r}) = \overline{u_i(\boldsymbol{x})u_j(\boldsymbol{x})\frac{\partial u_k}{\partial x_k}(\boldsymbol{x}+\boldsymbol{r})} = 0$$

$$R_{i,j,k}(\boldsymbol{r}) = A(r)\frac{r_i r_j r_k}{r^3} + B(r)\delta_{i,j}\frac{r_k}{r} + C(r)\left(\delta_{i,k}\frac{r_j}{r} + \delta_{j,k}\frac{r_i}{r}\right)$$

$$R_{i,j,k}(\boldsymbol{r}) = -R_{i,j,k}(-\boldsymbol{r}) \qquad R_{i,j,k}(0) = 0$$

92

Incompressibility

$$\frac{\partial}{\partial r_k}R_{i,j,k}(\boldsymbol{r}) = \left[\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{2}{r}\right)A + 2\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r}\right)C\right]\frac{r_i r_j}{r^2}$$

$$+ \left[\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{2}{r}\right)B + \frac{2}{r}C\right]\delta_{i,j}$$

$$= 0$$

$$C = -B - \frac{r}{2}\frac{\mathrm{d}B}{\mathrm{d}r}$$

$$A = -B + r\frac{\mathrm{d}B}{\mathrm{d}r}$$

93

3rd-order longitudinal correlation function

$$h(r) = \overline{u_L^2(\boldsymbol{x})u_L(\boldsymbol{x}+\boldsymbol{r})}/U^3$$

Longitudinal velocity correlation

$$U^3 h(r) = R_{i,j,k}\frac{r_i r_j r_k}{r^3} = -2B(r)$$

Isotropy

$$R_{1,1,1}(r_1\boldsymbol{e}_1) = -R_{1,1,1}(-r_1\boldsymbol{e}_1)$$

$$\frac{\partial^2}{\partial r_1^2}R_{1,1,1}(r_1\boldsymbol{e}_1) = -\frac{\partial^2}{\partial r_1^2}R_{1,1,1}(-r_1\boldsymbol{e}_1)$$

$$\frac{\partial^2}{\partial r_1^2}R_{1,1,1}(0) = -\frac{\partial^2}{\partial r_1^2}R_{1,1,1}(0) = 0$$

94

Homogeneity

$$\frac{\partial}{\partial r_1}R_{1,1,1}(0) = \overline{u_1^2\frac{\partial u_1}{\partial x_1}} = \frac{1}{3}\frac{\partial}{\partial x_1}\overline{u_1^3} = 0$$

$$\frac{\partial^3}{\partial r_1^3}R_{1,1,1}(0) = \overline{u_1^2\frac{\partial^3 u_1}{\partial x_1^3}}$$

$$= \frac{\partial}{\partial x_1}\overline{u_1^2\frac{\partial^2 u_1}{\partial x_1^2}} - \overline{\frac{\partial u_1^2}{\partial x_1}\frac{\partial^2 u_1}{\partial x_1^2}}$$

$$= -2\overline{u_1\frac{\partial u_1}{\partial x_1}\frac{\partial^2 u_1}{\partial x_1^2}} = -\overline{u_1\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}\right)^2}$$

$$= -\frac{\partial}{\partial x_1}\overline{u_1\left(\frac{\partial u_1}{\partial x_1}\right)^2} + \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3}$$

$$= \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3}$$

95

Longitudinal velocity correlation

$$U^3 h(r) = \frac{1}{6}\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3} r^3 + \cdots$$

2nd-order velocity structure tensor

$$B_{i,j}(\boldsymbol{r}) = \overline{\delta u_i(\boldsymbol{r})\delta u_j(\boldsymbol{r})}$$

$$= 2[R_{i,j}(0) - R_{i,j}(\boldsymbol{r})]$$

$$= S_2(r)\frac{r_i r_j}{r^2} + U_2(r)\left(\delta_{i,j} - \frac{r_i r_j}{r^2}\right)$$

$$\delta u_i(\boldsymbol{r}) = u_i(\boldsymbol{x}+\boldsymbol{r}) - u_i(\boldsymbol{x})$$

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2nd-order longitudinal velocity structure function

$$S_2(r) = \overline{[u_L(\boldsymbol{x}+\boldsymbol{r}) - u_L(\boldsymbol{x})]^2}$$
$$= 2U^2[1 - f(r)]$$

2nd-order lateral velocity structure function

$$U_2(r) = \overline{[u_N(\boldsymbol{x}+\boldsymbol{r}) - u_N(\boldsymbol{x})]^2}$$
$$= 2U^2\left[1 - f(r) - \frac{r}{2}\frac{\mathrm{d}f}{\mathrm{d}r}\right]$$
$$= S_2(r) + \frac{r}{2}\frac{\mathrm{d}S_2}{\mathrm{d}r}$$

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3rd-order velocity structure tensor

$$B_{i,j,k}(\boldsymbol{r}) = \overline{\delta u_i(\boldsymbol{r})\delta u_j(\boldsymbol{r})\delta u_k(\boldsymbol{r})}$$
$$= 2[R_{i,j,k}(\boldsymbol{r}) + R_{k,i,j}(\boldsymbol{r}) + R_{j,k,i}(\boldsymbol{r})]$$
$$= U^3\Big[3\left(h - r\frac{\mathrm{d}h}{\mathrm{d}r}\right)\frac{r_i r_j r_k}{r^3}$$
$$+ \left(h + r\frac{\mathrm{d}h}{\mathrm{d}r}\right)\left(\delta_{i,j}\frac{r_k}{r} + \delta_{j,k}\frac{r_i}{r} + \delta_{k,i}\frac{r_j}{r}\right)\Big]$$

3rd-order longitudinal velocity structure function

$$S_3(r) = \overline{[u_L(\boldsymbol{x}+\boldsymbol{r}) - u_L(\boldsymbol{x})]^3}$$
$$= B_{i,j,k}(\boldsymbol{r})\frac{r_i r_j r_k}{r^3}$$
$$= 6U^3 h(r)$$

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Velocity correlation equation

$$u_i = u_i(\boldsymbol{x}, t) \qquad u_i' = u_i(\boldsymbol{x}', t)$$
$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{r}$$
$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k}u_i u_k = -\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2}{\partial x_k \partial x_k}u_i + f_i$$
$$\frac{\partial u_j'}{\partial t} + \frac{\partial}{\partial x_k'}u_j' u_k' = -\frac{\partial p'}{\partial x_j'} + \nu\frac{\partial^2}{\partial x_k' \partial x_k'}u_j' + f_j'$$
$$\frac{\partial}{\partial t}\overline{u_i u_j'} = -\frac{\partial}{\partial r_k}\left(\overline{u_i u_k u_j'} - \overline{u_i u_k' u_j'}\right)$$
$$-\frac{\partial}{\partial r_i}\overline{p u_j'} + \frac{\partial}{\partial r_j}\overline{p' u_i} + 2\nu\frac{\partial^2}{\partial r_k \partial r_k}\overline{u_i u_j'} + \overline{f_i u_j'} + \overline{f_j' u_i}$$

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$$\frac{\partial}{\partial t}R_{i,j}(\boldsymbol{r},t) = \frac{\partial}{\partial r_k}\left[R_{i,k,j}(\boldsymbol{r},t) + R_{j,k,i}(\boldsymbol{r},t)\right]$$
$$-\frac{\partial}{\partial r_i}\overline{p u_j'} + \frac{\partial}{\partial r_j}\overline{p' u_i} + 2\nu\frac{\partial^2}{\partial r_k \partial r_k}R_{i,j}(\boldsymbol{r},t) + F_{i,j}$$

Pressure-velocity correlation

$$\overline{p u_j'} = A(r,t)\frac{r_j}{r}$$
$$\frac{\partial}{\partial r_j}\overline{p u_j'} = \frac{\partial}{\partial r_j}\left(A\frac{r_j}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A) = 0$$
$$A(r,t) = \frac{C}{r^2}$$
$$\overline{p u_j'} = 0 \quad (C = 0)$$

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$$\left(\frac{\partial}{\partial t} - 2\nu\frac{\partial^2}{\partial r_k \partial r_k}\right)R_{i,j}(\boldsymbol{r},t)$$
$$= \frac{\partial}{\partial r_k}\left[R_{i,k,j}(\boldsymbol{r},t) + R_{j,k,i}(\boldsymbol{r},t)\right] + F_{i,j}$$

3rd order velocity correlation terms

$$\frac{\partial}{\partial r_k}\left[R_{i,k,j}(\boldsymbol{r},t) + R_{j,k,i}(\boldsymbol{r},t)\right]\Big|_{r=0}$$
$$= \overline{u_i u_k\frac{\partial u_j}{\partial x_k}} + \overline{u_j u_k\frac{\partial u_i}{\partial x_k}}$$
$$= \frac{\partial}{\partial x_k}\overline{u_i u_j u_k} = 0$$

101

Isotropic form $i = j$

$$\left(3 + r\frac{\partial}{\partial r}\right)\Big[\frac{\partial}{\partial t}U^2 f - 2\nu U^2\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)\frac{\partial f}{\partial r}$$
$$-U^3\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)h - F(r,t)\Big] = 0$$
$$F(r,t) = \frac{1}{r^3}\int_0^r r'^2 F_{i,i}(r',t)\mathrm{d}r'$$

Karman—Howarth equation

$$\left[\frac{\partial}{\partial t} - 2\nu\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)\frac{\partial}{\partial r}\right]U^2 f = \left(\frac{\partial}{\partial r} + \frac{4}{r}\right)U^3 h + F$$

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Energy dissipation

Decaying turbulence $F(r,t) = 0$

$$\frac{\partial}{\partial t}U^2 = 10\nu U^2 \left.\frac{\partial^2 f}{\partial r^2}\right|_{r=0}$$

$$\bar{\epsilon} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{3}{2}U^2\right) = 15\nu\left(\frac{U}{\lambda}\right)^2$$

Stationary turbulence $\partial/\partial t = 0$

$$F(r=0) = -10\nu U^2 \left.\frac{\partial^2 f}{\partial r^2}\right|_{r=0}$$

$$\bar{\epsilon} = \tfrac{3}{2}F(r=0) = 15\nu\left(\frac{U}{\lambda}\right)^2$$

103

Integral scale

$$\frac{U^3}{L} \sim \nu\left(\frac{U}{\lambda}\right)^2$$

$$\frac{L}{\lambda} \sim \frac{U\lambda}{\nu} = R_\lambda$$

$$Re = \frac{UL}{\nu} \sim \frac{U\lambda}{\nu}\frac{L}{\lambda} = R_\lambda^2$$

Kolmogorov scale

$$\eta = (\nu^3/\bar{\epsilon})^{1/4}$$

$$\frac{\eta}{\lambda} \sim R_\lambda^{-1/2}$$

$$\frac{\eta}{L} \sim R_\lambda^{-3/2} \sim Re^{-3/4}$$

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2nd, 3rd order longitudinal velocity correlation

$$U^2 f(r,t) = U^2 - \tfrac{1}{2}S_2(r,t)$$

$$U^3 h(r,t) = \tfrac{1}{6}S_3(r,t)$$

Karman—Howarth equation

$$\frac{\mathrm{d}U^2}{\mathrm{d}t} - \frac{1}{2}\frac{\partial S_2}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\left(\tfrac{1}{6}S_3 - \nu\frac{\partial S_2}{\partial r}\right)\right] + F$$

Small scale $r \ll L$

$$\tfrac{1}{6}S_3 - \nu\frac{\partial S_2}{\partial r} = -\tfrac{2}{15}\bar{\epsilon}r$$

105

Inertial range $\eta \ll r \ll L$

$$\tfrac{1}{6}S_3 = -\tfrac{2}{15}\bar{\epsilon}r$$

Kolmogorov 4/5 law

$$S_3 = \overline{[u_L(\boldsymbol{x}+\boldsymbol{r},t) - u_L(\boldsymbol{x},t)]^3} = -\tfrac{4}{5}\bar{\epsilon}r$$

$$U^3 h = \overline{u_L^2(\boldsymbol{x},t)u_L(\boldsymbol{x}+\boldsymbol{r},t)} = -\tfrac{2}{15}\bar{\epsilon}r$$

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3.2 Universal statistical laws

Andrei N. Kolmogorov
(1903 – 1987)
Kolmogorov similarity law
(1941)

Ludwig Prandtl
(1875 – 1953)
Prandtl wall law
(1932)

107

Fourier expansion L^3 Periodic box

$$\boldsymbol{u}(\boldsymbol{x},t) = \left(\frac{2\pi}{L}\right)^3 \sum_{\boldsymbol{k}} \tilde{\boldsymbol{u}}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$L \to \infty \quad \sum_{\boldsymbol{k}}\left(\frac{2\pi}{L}\right)^3 \to \int \mathrm{d}\boldsymbol{k}$$

Wavenumber vector

$$\boldsymbol{k} = \frac{2\pi}{L}\boldsymbol{m} \quad \boldsymbol{m} = (m_1, m_2, m_3)^t \quad m_i : \text{integer}$$

Fourier coefficient

$$\tilde{\boldsymbol{u}}(\boldsymbol{k},t) = \left(\frac{1}{2\pi}\right)^3 \int \boldsymbol{u}(\boldsymbol{x},t)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{x}$$

$$\tilde{\boldsymbol{u}}(-\boldsymbol{k},t) = \tilde{\boldsymbol{u}}^*(\boldsymbol{k},t)$$

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Energy spectral tensor

$$\Phi_{i,j}(\boldsymbol{k},t)=\left(\frac{1}{2\pi}\right)^3\int R_{i,j}(\boldsymbol{r},t)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$$

$$R_{i,j}(\boldsymbol{r},t)=\int \Phi_{i,j}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{k}$$

Energy spectral function

$$\frac{1}{2}\overline{|\boldsymbol{u}|^2}=\frac{3}{2}U^2=\frac{1}{2}R_{i,i}(0,t)=\frac{1}{2}\int \Phi_{i,i}(\boldsymbol{k},t)\mathrm{d}\boldsymbol{k}$$

$$=\int_0^\infty 2\pi k^2\Phi_{i,i}(k,t)\mathrm{d}k \qquad k=|\boldsymbol{k}|$$

$$=\int_0^\infty E(k,t)\mathrm{d}k$$

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Vorticity correlation tensor

$$R^\omega_{i,j}(\boldsymbol{r})=\overline{\omega_i(\boldsymbol{x}+\boldsymbol{r})\omega_j(\boldsymbol{x})}=\overline{\omega_i(\boldsymbol{x})\omega_j(\boldsymbol{x}')}$$

$$R^\omega_{i,i}(\boldsymbol{r})=\varepsilon_{i,p,q}\varepsilon_{i,l,m}\overline{\frac{\partial u_q}{\partial x_p}\frac{\partial u'_m}{\partial x'_l}}$$

$$=\overline{\frac{\partial u_m}{\partial x_l}\frac{\partial u'_m}{\partial x'_l}}-\overline{\frac{\partial u_l}{\partial x_m}\frac{\partial u'_m}{\partial x'_l}}$$

$$=\frac{\partial^2}{\partial x_l\partial x'_l}R_{m,m}-\frac{\partial^2}{\partial x_m\partial x'_l}R_{l,m}$$

$$=-\frac{\partial^2}{\partial r_l^2}R_{m,m}+\frac{\partial^2}{\partial r_m\partial r_l}R_{l,m}$$

$$=-\frac{\partial^2}{\partial r_l^2}R_{m,m}$$

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Enstrophy spectral tensor

$$\Phi^\omega_{i,j}(\boldsymbol{k},t)=\left(\frac{1}{2\pi}\right)^3\int R^\omega_{i,j}(\boldsymbol{r},t)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$$

$$R^\omega_{i,j}(\boldsymbol{r},t)=\int \Phi^\omega_{i,j}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{k}$$

Enstrophy spectral function

$$R^\omega_{i,i}(\boldsymbol{r},t)=-\frac{\partial^2}{\partial r_l^2}\int \Phi_{i,i}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{k}$$

$$=\int k^2\Phi_{i,i}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{k}$$

$$\frac{1}{2}\overline{|\boldsymbol{\omega}|^2}=\frac{1}{2}R^\omega_{i,i}(0,t)=\frac{1}{2}\int k^2\Phi_{i,i}(\boldsymbol{k},t)\mathrm{d}\boldsymbol{k}$$

$$=\int_0^\infty 2\pi k^4\Phi_{i,i}(k,t)\mathrm{d}k=\int_0^\infty k^2E(k,t)\mathrm{d}k$$

111

Energy dissipation

$$2\nu\frac{\partial^2}{\partial r_k^2}R_{i,i}=-2\nu R^\omega_{i,i}$$

$$2\nu\left.\frac{\partial^2}{\partial r_k^2}R_{i,i}\right|_{r=0}=-2\nu R^\omega_{i,i}(0,t)=-2\nu\overline{|\boldsymbol{\omega}|^2}$$

$$3\frac{\mathrm{d}U^2}{\mathrm{d}t}+2\nu\overline{|\boldsymbol{\omega}|^2}=F_{i,i}(r=0)$$

decaying $\quad \overline{\epsilon}=-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2}U^2\right)=\nu\overline{|\boldsymbol{\omega}|^2}$

stationary $\quad \overline{\epsilon}=\frac{1}{2}F_{i,i}(r=0)=\nu\overline{|\boldsymbol{\omega}|^2}$

Energy dissipation spectrum $\quad 2\nu k^2E(k,t)$

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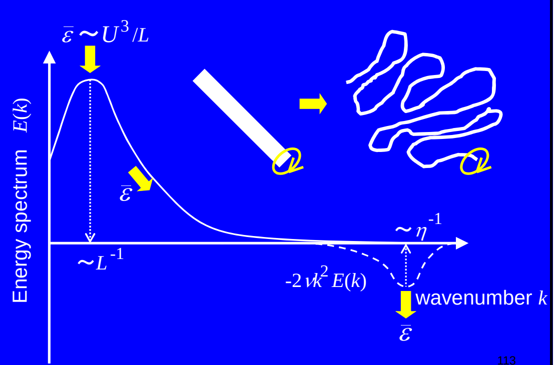
Energy cascade



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Kolmogorov similarity law

Energy dissipation

$$\overline{\epsilon}=\frac{\nu}{2}\overline{e_{i,j}e_{i,j}} \qquad e_{i,j}=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}$$

Kolmogorov scale

$$\eta=(\nu^3/\overline{\epsilon})^{1/4},\ t_K=(\nu/\overline{\epsilon})^{1/2},\ u_K=(\nu\overline{\epsilon})^{1/4}$$

Energy spectrum function (Kolmogorov 1st hypothesis)

$$E(k)=(\nu^5\overline{\epsilon})^{1/4}f[k(\nu^3/\overline{\epsilon})^{1/4}]$$

-5/3 power law (Kolmogorov 2nd hypothesis)

$$E(k)=C_K\overline{\epsilon}^{2/3}k^{-5/3} \qquad C_K=1.4-1.8$$

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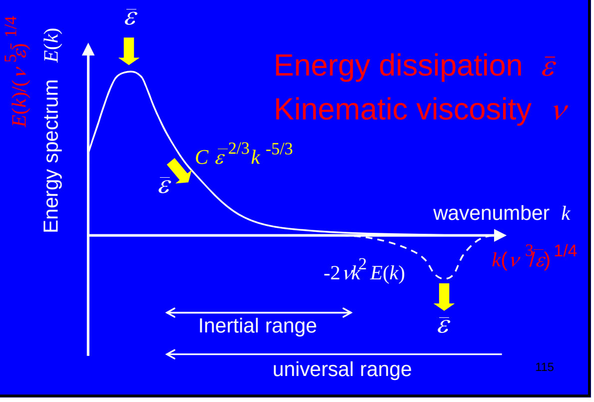
Universal energy spectrum



Obukov cascade

Inertial range $\bar{\epsilon},\ r$

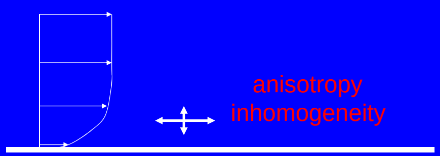
$$\delta u(r) \sim (\bar{\epsilon} r)^{1/3}$$

velocity $$\frac{\delta u(r)}{\delta u(r')} \sim \left(\frac{r}{r'}\right)^{1/3}$$ Scale invariance $\alpha = 1$

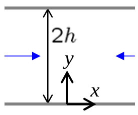
vorticity $$\frac{\omega(r)}{\omega(r')} \sim \left(\frac{r}{r'}\right)^{-2/3}$$

Intermittency Deviation from $\overline{[\delta u(r)]^p}/U^p \sim (r/L)^{p/3}$

Near-wall turbulence



- Boundary layer
- Channel flow (Couette, Poiseuille)
- Pipe flow (Hagen-Poiseuille)



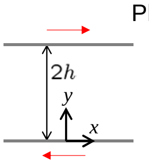
Plane Poiseuille flow

$$\bar{p} + \overline{v'^2} = \bar{p}|_w$$

$$\nu\frac{d\bar{u}}{dy} - \overline{u'v'} = \nu\left.\frac{d\bar{u}}{dy}\right|_w - \frac{\partial\bar{p}}{\partial x}y$$

Plane Couette flow $(\partial\bar{p}/\partial x = 0)$

$$\nu\frac{d\bar{u}}{dy} - \overline{u'v'} = \nu\left.\frac{d\bar{u}}{dy}\right|_w$$



Plane Poiseuille flow

$$\bar{p} + \overline{v'^2} = \bar{p}|_w$$

$$\nu\frac{d\bar{u}}{dy} - \overline{u'v'} = \nu\left.\frac{d\bar{u}}{dy}\right|_w - \frac{\partial\bar{p}}{\partial x}y$$

Plane Couette flow $(\partial\bar{p}/\partial x = 0)$

$$\nu\frac{d\bar{u}}{dy} - \overline{u'v'} = \nu\left.\frac{d\bar{u}}{dy}\right|_w$$

Friction velocity

$$u_\tau = \sqrt{\nu\left.\frac{d\bar{u}}{dy}\right|_w}$$

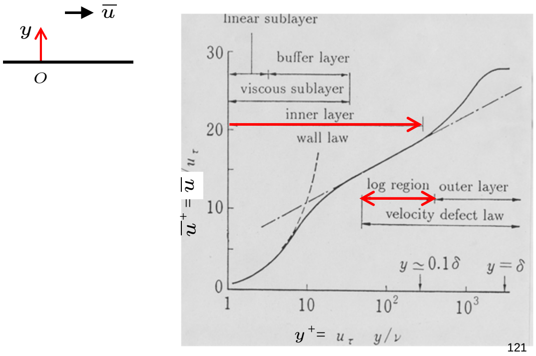
Prandtl wall law (near-wall region $y/h \ll 1$)

$$\nu\frac{d\bar{u}}{dy} - \overline{u'v'} = u_\tau^2$$

$$\bar{u} = u_\tau f\left(\frac{u_\tau y}{\nu}\right)$$

$$\frac{\bar{u}}{u_\tau} = \frac{u_\tau y}{\nu} \qquad (u_\tau y/\nu \sim 1)$$

Mean velocity



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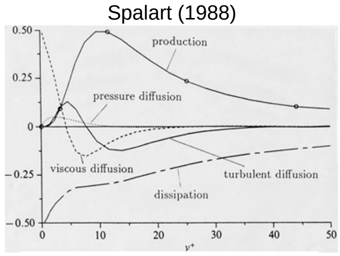
Energy budget

Spalart (1988)



$$-\frac{\mathrm{d}\phi_2}{\mathrm{d}y} + \nu\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\overline{v_i'v_i'}/2\right) - \overline{u'v'}\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - \epsilon = 0$$

× 2

$$\phi_2 = \overline{v'\left(p' + v_i'v_i'/2\right)}, \quad \epsilon = \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

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Logarithmic velocity (Townsend 1976)

Turbulence energy production

$$u_\tau^2\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}$$

Energy dissipation ∼ Energy input

$$\epsilon \sim \frac{u_\tau^3}{\kappa y}$$

Energy production ∼ Energy dissipation

$$\frac{\mathrm{d}\overline{u}^+}{\mathrm{d}y^+} \sim \frac{1}{\kappa y^+} \rightarrow \overline{u}^+ \sim \frac{1}{\kappa}\ln y^+ + C$$

$\kappa = 0.41$, C depends on flow

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